Solutions to Review Problems I

1. \( y^2 = 2\tan^{-1}(t) + 2 \).

2. \( y(t) = -\frac{1}{3} + t + e^{-2t} + Ce^{-3t} \).

3. \( \ln y + \frac{y^2}{2} = -\cos x + \frac{3}{2} \).

4. \( M = x^4 + \frac{2y}{x} \) and \( N = y^3 + 2\ln x \). Hence
   \[ M_y = \frac{2}{x}, \quad N_x = 2\left(\frac{1}{x}\right). \]
   Thus the equation is exact. The general solution is given by
   \[ \frac{x^5}{5} + 2y\ln x + \frac{y^4}{4} = C. \]

5. The exactness implies \( (f(x)y)_y = (x + f(x))_x \), i.e.
   \[ f(x) = 1 + f'(x). \]
   Hence \( f(x) = 1 + Ce^x \).

6. (a) \( r = -\frac{\ln 2}{100} \)
   (b) \( r = \frac{\ln 2}{200} \).

7. \( y_1 = a \) and \( y_2 = b \) are two equilibrium solutions of the equation. \( y_1 = a \) is stable, and \( y_2 = b \) is unstable. The solution to the initial value problem is:
   \[ \ln \left| \frac{y - b}{y - a} \right| = (b - a)t + \ln \left| \frac{y_0 - b}{y_0 - a} \right|. \]

8. The equation is give by
   \[ \frac{dM}{dt} = 0.09M - 800 \times 12; \quad M(0) = M_0. \]
   Then the equation has its solution
   \[ M(t) = \frac{960000}{9}(1 - e^{0.09t}) + M_0e^{0.09t}. \]
If $M(20) = 0$, then we can solve for $M_0$.

9. Set $Z(t)$ to be the function of pollutant at time $t$. Then

$$\frac{dZ}{dt} = 0.4 - \frac{Z}{100} \times 4; Z(0) = 50.$$ 

Solving the equation gives

$$Z(t) = 10 + 40e^{-\frac{t}{25}}.$$ 

10. $y_c = c_1e^{\frac{t}{2}} + c_2e^{\frac{t}{3}}$, where $c_1 = -8$ and $c_2 = 12$. 