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SUBOPTIMAL AND OPTIMAL CONVERGENCE IN MIXED FINITE ELEMENT METHODS*

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Abstract. An elliptic partial differential equation may be formulated in different but equivalent ways, and the mixed finite element methods derived from these formulations have different properties. We give general error estimates for two such methods, which are always optimal for the Raviart–Thomas elements, but which are suboptimal for the Brezzi–Douglas–Marini elements in one of the methods. Computational experiments show that this suboptimal estimate is sharp.

Key words. mixed finite element methods, suboptimal convergence, optimal convergence

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1. Introduction.

In this paper we consider two mixed finite element methods corresponding to two equivalent ways of formulating a general second-order elliptic scalar problem for $u(x)$, where $x \in \Omega$, Ω is a bounded domain in $\mathbb{R}^n$, and $n \geq 2$. Written in "divergence" form, the problem is to find u satisfying

$$- \sum_{i,j=1}^{n} \frac{\partial}{\partial x_j} \left(a_{ij}(x) \frac{\partial u}{\partial x_i} \right) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + c(x)u = f \text{ in } \Omega,$$
$$u = g \text{ on } \partial\Omega,$$

or, with the matrix $A = [a_{ij}]$ and the vector $\vec{b} = [b_i]$,

$$\begin{aligned} -\operatorname{div}(A\nabla u) + \vec{b}.\nabla u + cu &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{1.1}$$

Under minimal smoothness assumptions on the coefficient $\vec{b}$, the problem may be equivalently formulated in the "conservation" (or "strong divergence") form

$$\begin{aligned} -\operatorname{div}(A\nabla u + \vec{b}u) + cu &= f \text{ in } \Omega, \\ u &= g \text{ on } \partial\Omega. \end{aligned} \tag{1.2}$$

We note here that, although (1.1) and (1.2) are equivalent forms, they are not the same differential equation (their weak forms are, in fact, adjoints in the case that A is symmetric and $g = 0$), and we shall treat them as different problems.

The main point of this investigation is to show that the convergence of the "natural" mixed finite element approximation to the vector variable $\tilde{p} = -(A\nabla u + \vec{b}u)$ in (1.2) may be of suboptimal order, while mixed methods based on (1.1) approximate the vector variable $\vec{p} = -A\nabla u$ to optimal order for all standard choices of element spaces. Suboptimal convergence in methods based on (1.2) may occur when

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the finite element spaces employed have a higher order of approximation for the vector variable $\tilde{p}$ than for the scalar variable u. This is, in particular, the case for the Brezzi–Douglas–Marini (BDM) family of elements for simplicial decompositions.

Mixed finite element methods for approximating solutions to (1.1) and (1.2) when the coefficient $\vec{b}$ is nonzero were introduced in [DR82]. Let $\vec{Q}_h \times V_h \subset H(\text{div};\Omega) \times L_2(\Omega)$ be a mixed approximating subspace. Then the mixed finite element method corresponding to the divergence form (1.1) is as follows: Find a pair $\{\vec{p}_h, u_h\} \in \vec{Q}_h \times V_h$ such that

$$\begin{aligned} (A^{-1}\vec{p}_h, \vec{q}_h) - (\operatorname{div}\vec{q}_h, u_h) &= -\langle g, \vec{q}_h \cdot \vec{n}\rangle, \\ (\operatorname{div}\vec{p}_h, v_h) - (\vec{b}.A^{-1}\vec{p}_h, v_h) + (cu_h, v_h) &= (f, v_h) \end{aligned} \tag{1.3}$$

for all $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$. Here $(\cdot,\cdot)$ denotes the $L_2(\Omega)$ or $[L_2(\Omega)]^n$ inner product, and $\langle\cdot,\cdot\rangle$ denotes the $L_2(\partial\Omega)$ inner product. Note that here the vector variable $\vec{p}_h$ approximates $\vec{p} = -A\nabla u$.

The corresponding mixed method for the conservation-form problem (1.2) is as follows: Find $\{\tilde{p}_h, u_h\} \in \vec{Q}_h \times V_h$ such that

$$\begin{aligned} (A^{-1}\tilde{p}_h, \vec{q}_h) - (\operatorname{div}\vec{q}_h, u_h) + (A^{-1}\vec{b}u_h, \vec{q}_h) &= -\langle g, \vec{q}_h \cdot \vec{n}\rangle, \\ (\operatorname{div}\tilde{p}_h, v_h) + (cu_h, v_h) &= (f, v_h) \end{aligned} \tag{1.4}$$

for all $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$. The vector variable $\tilde{p}_h$ now approximates $\tilde{p} = -(A\nabla u + \vec{b}u)$.

We will take $\vec{Q}_h \times V_h$ to be any of a number of families of elements defined on decompositions of Ω into simplices or quadrilaterals of maximal diameter h (see section 2.3 for precise assumptions). The BDM family of elements for triangular (in the case $n = 2$) or simplicial (in the case $n = 3$) decompositions, members of which are denoted BDM_k, will be of special interest. In the BDM family, introduced in [BDM85], we have $\vec{Q}_h = [P_k]^n$ and $V_h = P_{k-1}$ on each element with conditions across element interfaces so that $\operatorname{div}\vec{q}_h \in L_2(\Omega)$ for $\vec{q}_h \in \vec{Q}_h$. A local interpolant Π_h is defined, and

$$||\vec{q} - \Pi_h\vec{q}||_{[L_2(\Omega)]^n} \leq Ch^{k+1}||\vec{q}||_{[H^{k+1}(\Omega)]^n}. \tag{1.5}$$

The local L_2-projection $P_h : L_2(\Omega) \to V_h$ is used as the scalar-field interpolant and satisfies

$$||v - P_h v||_{L_2(\Omega)} \leq Ch^k||v||_{H^k(\Omega)}. \tag{1.6}$$

Thus, the vector finite element space $\vec{Q}_h$ is able to approximate to one order higher than the scalar space V_h. In contrast to the BDM elements, the commonly used Raviart–Thomas spaces approximate the vector and scalar fields to the same order. More important, the BDM elements approximate the vector field to a given order with fewer total degrees of freedom than are required by the Raviart–Thomas elements. The efficiency with which the BDM elements approximate the vector field is advantageous since mixed methods are designed to approximate the vector variable well.

Existence, uniqueness, and optimal order global L_2 estimates for both the scalar and vector variables were proven for (1.3) and (1.4) in [DR82] for the case where $\vec{Q}_h \times V_h$ is taken to be one of the Raviart–Thomas ($n = 2$) or Raviart–Thomas–Nédélec ($n = 3$) family of spaces and where h is small enough. Global L_∞ results and global L_2 and negative-norm results for the conservation-form method were also given in [GN88] and [DR85], respectively. However, these results are not valid when the BDM_k spaces are used, and similar results for these spaces do not exist in the literature. In this paper, we investigate the order of convergence of methods (1.3) and (1.4) via both theoretical and computational means.

Our main focus will be on estimates for the errors $\vec{p}-\vec{p}_h$ and $\tilde{p}-\tilde{p}_h$ in the vector variables. In contrast to [DR82] and [DR85], we shall also pay particular attention to the structure of the error bounds. For the sake of comparison, we first state results for the well-studied case (see, e.g., [FO80] and [Du88]) where both $\vec{b}$ and c are zero.

THEOREM 1.1. *Assume that both $\vec{b}$ and c are zero in* (1.1) *and* (1.3) *and that the assumptions of section* 2 *concerning the divergence-form error equations and the finite element spaces are satisfied. Then there exists a constant C independent of h, u, $\vec{p}$, f, and g such that*

$$\|\vec{p}-\vec{p}_h\|_{[L_2(\Omega)]^n} \le C\|\vec{p}-\Pi_h\vec{p}\|_{[L_2(\Omega)]^n}.$$

For the divergence-form problem (1.3), we prove a new error estimate which is valid under general abstract assumptions and which yields an optimal order of convergence for all element families normally used in the context of mixed methods for elliptic problems (including BDM_k).

THEOREM 1.2. *Let the assumptions given in section* 2 *for the divergence-form problem be satisfied. Then for h small enough, the solution $\{\vec{p}_h, u_h\}$ to* (1.3) *exists and is unique, and there exists a constant C independent of h, u, $\vec{p}$, f, and g such that*

$$\|\vec{p}-\vec{p}_h\|_{[L_2(\Omega)]^n} \le C(\|\vec{p}-\Pi_h\vec{p}\|_{[L_2(\Omega)]^n} + h\|u-P_hu\|_{L_2(\Omega)}). \tag{1.7}$$

Remark 1.3. It is possible to sharpen the estimate (1.7) when the coefficient c is constant on each element of the mesh. In particular, it then can be shown that for h small enough,

$$\|\vec{p}-\vec{p}_h\|_{[L_2(\Omega)]^n} \le C\|\vec{p}-\Pi_h\vec{p}\|_{[L_2(\Omega)]^n}.$$

Finally, we prove the following estimate for the conservation-form problem (1.4).

THEOREM 1.4. *Let the assumptions given in section* 2 *for the conservation-form problem be satisfied. Then for h small enough, the solution $\{\tilde{p}_h, u_h\}$ to* (1.4) *exists and is unique, and there exists a constant C independent of h, u, $\tilde{p}$, f, and g such that*

$$\|\tilde{p}-\tilde{p}_h\|_{[L_2(\Omega)]^n} \le C(\|\tilde{p}-\Pi_h\tilde{p}\|_{[L_2(\Omega)]^n} + \|u-P_hu\|_{L_2(\Omega)}). \tag{1.8}$$

In the progression from Theorem 1.1 to Theorem 1.4, our error estimates exhibit a strengthening of the coupling of the vector variable $\vec{p}_h$ or $\tilde{p}_h$ to the scalar variable u_h. In Theorem 1.1, the error for the vector variable is not coupled to the approximation error for the scalar variable, as the bound given for $\|\vec{p}-\vec{p}_h\|_{[L_2(\Omega)]^n}$ is independent of u. The estimate (1.7) for the divergence-form problem indicates a weak coupling of $\vec{p}-\vec{p}_h$ to $u-P_hu$ in the presence of lower-order terms. That is, the bound for $\|\vec{p}-\vec{p}_h\|_{[L_2(\Omega)]^n}$ is dependent on u, but only via an approximation term $\|u-P_hu\|_{L_2(\Omega)}$ multiplied by a factor of h. The estimate (1.8) indicates that when the vector variable $\tilde{p}$ is directly related to u and not just to its first derivatives, a strong coupling of $\tilde{p}-\tilde{p}_h$ to $u-P_hu$ results. That is, the bound for $\|\tilde{p}-\tilde{p}_h\|_{[L_2(\Omega)]^n}$ is dependent on $\|u-P_hu\|_{L_2(\Omega)}$ but now without a factor of h multiplying the latter term.

We next note that combining the approximation estimates (1.5) and (1.6) with the error bound (1.8) yields a suboptimal order of convergence in the vector variable for the conservation-form method (1.4) when the BDM_k spaces are employed. We are thus naturally led to question whether (1.8) is sharp. In section 4 we present numerical examples for which the convergence of $\tilde{p}_h$ to $\tilde{p}$ is of suboptimal order. In particular, we compute estimated rates of convergence for problems using the BDM_1 and BDM_2 elements in $\mathbb{R}^2$. These spaces are able to approximate the vector variable to orders 2 and 3, but we compute rates of convergence which clearly tend to 1 and 2, respectively, as the mesh is subdivided. That is, the convergence of $\tilde{p}_h$ to $\tilde{p}$ is of suboptimal order, and the estimate (1.8) is indeed sharp.

We finally comment on the relationship between our work and previous work. In [Ch94], Chen discussed the use of BDM elements in mixed approximations to solutions of certain nonlinear equations. In that paper, he presented two computational examples in which the vector variable was poorly approximated when using the BDM_1 elements in a standard mixed method on a mesh of fixed size. Chen also conjectured that the rate of convergence of the mixed approximation to the vector variable may be suboptimal when using the BDM elements with standard mixed methods for nonlinear equations, and he further suggested that such suboptimal rates of convergence may be due to a coupling between the vector and scalar variables. In [Ch94], [Ch98a], and [Ch98b], he gave expanded mixed methods for both linear and nonlinear problems which lead to optimal rates of convergence when the BDM elements are used.

2. Preliminaries and assumptions.

In this section, we outline the assumptions under which we shall prove our results, and we also describe the BDM family of elements used in our computations.

2.1. Assumptions for the partial differential equation.

We assume that the matrix of coefficients $A = [a_{ij}(x)]$ satisfies the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq C_{ell}|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad x \in \Omega,$$

where $C_{ell} > 0$. We also assume that the coefficients a_{ij} and b_i are Lipschitz continuous and that c is bounded and Lipschitz continuous on each element of the mesh.

We do not require coercivity of the bilinear forms corresponding to (1.1) and (1.2). Instead, we make weaker assumptions concerning existence and regularity of solutions to (1.1) and (1.2). We denote by $H_0^1(\Omega)$ members of $H^1(\Omega)$ with vanishing trace and denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$. Then we assume that if $f \in H^{-1}(\Omega)$ and $g = 0$, (1.1) and (1.2) have unique solutions. Thus $u \in H_0^1(\Omega)$ solving (1.1) or (1.2) satisfies

$$\|u\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{2.1}$$

If $f \in L_2(\Omega)$ and $\psi \in H_0^1(\Omega)$ weakly satisfy $-\operatorname{div}(A\nabla\psi) = f$, then we further require the regularity estimate

$$\|\psi\|_{H^2(\Omega)} \leq C\|f\|_{L_2(\Omega)}. \tag{2.2}$$

We note that Lipschitz continuity of the coefficients a_i guarantees that (2.2) will hold whenever $\partial\Omega$ is C^2 or Ω is convex; cf. [Gr85, Thms. 2.3.3.2 and 3.2.1.2].

In our proofs, we shall make use of uniqueness and regularity properties of the formal adjoints in $H_0^1(\Omega)$ to (1.1) and (1.2). These properties are equivalent to those stated above for (1.1) and (1.2), but we collect them here for convenience. For Theorem 1.2, we require existence and uniqueness of solutions to

$$\begin{aligned} -\operatorname{div}(A^*\nabla\phi + \vec{b}\phi) + c\phi &= f \text{ in } \Omega, \\ \phi &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{2.3}$$

and we also require that the H^2 regularity estimate

$$\|\phi\|_{H^2(\Omega)} \le C\|f\|_{L_2(\Omega)} \tag{2.4}$$

holds. The latter estimate follows from (2.1) and (2.2). Let $f = \operatorname{div}\vec{z}$, where $\vec{z} \in H(\operatorname{div};\Omega) = \{\vec{q} \in [L_2(\Omega)]^n \text{ s.t. } \operatorname{div}\vec{q} = \sum_{i=1}^n \frac{d\vec{q}_i}{dx_i} \in L_2(\Omega)\}$. We shall also require the energy-type estimate

$$\|\phi\|_{H^1(\Omega)} \le C\|\vec{z}\|_{[L_2(\Omega)]^n}, \tag{2.5}$$

which is a trivial consequence of (2.1). In our proof of Theorem 1.4, we require that solutions ϕ of

$$\begin{aligned} -\operatorname{div}(A^*\nabla\phi) + \vec{b}.\nabla\phi + c\phi &= f \text{ in } \Omega, \\ \phi &= 0 \text{ on } \partial\Omega, \end{aligned}$$

exist, be unique, and satisfy the H^2 regularity estimate (2.4).

2.2. Error equations. Theorems 1.1 and 1.2 do not require that u be an actual solution of (1.1) but only that the pair $\{\vec{p}-\vec{p}_h, u-u_h\}$ satisfy the divergence-form error equations

$$(A^{-1}(\vec{p}-\vec{p}_h), \vec{q}_h) - (\operatorname{div}\vec{q}_h, u-u_h) = 0, \tag{2.6a}$$

$$(\operatorname{div}(\vec{p}-\vec{p}_h), v_h) - (\vec{b}.A^{-1}(\vec{p}-\vec{p}_h), v_h) + (c(u-u_h), v_h) = 0 \tag{2.6b}$$

for $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$. In the case of Theorem 1.1 here of course both $\vec{b}$ and c are zero.

Similarly, for Theorem 1.4 we require that the pair $\{\tilde{p}-\tilde{p}_h, u-u_h\}$ satisfy the conservation-form error equations

$$(A^{-1}(\tilde{p}-\tilde{p}_h), \vec{q}_h) - (\operatorname{div}\vec{q}_h, u-u_h) + (A^{-1}\vec{b}(u-u_h), \vec{q}_h) = 0, \tag{2.7a}$$

$$(\operatorname{div}(\tilde{p}-\tilde{p}_h), v_h) + (c(u-u_h), v_h) = 0 \tag{2.7b}$$

for all $\{\vec{q}_h, v_h\} \in \vec{Q}_h \times V_h$.

In other words, we shall assume that all integrals are exact, including the boundary integrals $\langle g, \vec{q}\cdot n\rangle$ and $\langle g, \vec{q}_h \cdot n\rangle$ in the original problems.

2.3. Assumptions for the finite element spaces. Let τ_h be a partition of Ω into triangles or rectangles (or their higher-dimensional analogues) of maximum diameter h. Boundary elements are allowed to have one curved edge as in, for example, [DR85] and [BDM85]. Let the finite element space $\vec{Q}_h \times V_h \subset H(\operatorname{div};\Omega) \times L_2(\Omega)$ be defined with respect to τ_h.

We first require that the commuting diagram property be satisfied. We let $W = H(\operatorname{div};\Omega) \cap [L^s(\Omega)]^n$ for some fixed $s > 2$ and let $P_h : L_2(\Omega) \to V_h$ be the L_2 projection. Then we require that there exist a projection operator $\Pi_h : W \to \vec{Q}_h$ such that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{\Pi_h} & \vec{Q}_h \\ \Big\downarrow \text{div} & & \Big\downarrow \text{div} \\ L_2(\Omega) & \xrightarrow{P_h} & V_h \end{array}$$

The commuting diagram property can also be stated in the form

$$\operatorname{div} \Pi_h \vec{q} = P_h \operatorname{div} \vec{q} \tag{2.8}$$

for $\vec{q} \in W$. We also assume that $\Pi_h \vec{q}_h = \vec{q}_h$ for $\vec{q}_h \in \vec{Q}_h$, that is, that Π_h is the identity on the finite element space $\vec{Q}_h$ (clearly, P_h is the identity on V_h).

Approximation assumptions are stated next. Since the proofs of Theorems 1.1, 1.2, and 1.4 use only first-order approximation properties, that is all we shall require for the present. For the vector finite element space $\vec{Q}_h$ with interpolant Π_h, we assume

$$\|\vec{q} - \Pi_h \vec{q}\|_{[L_2(\Omega)]^n} \leq Ch\|\vec{q}\|_{[H^1(\Omega)]^n} \tag{2.9}$$

for all $\vec{q} \in [H^1(\Omega)]^n$. For the scalar field space V_h with L_2-projection P_h, we assume

$$\|v - P_h v\|_{L_2(\Omega)} \leq Ch\|v\|_{H^1(\Omega)} \tag{2.10}$$

for all $v \in H^1(\Omega)$. We note that these assumptions place very few restrictions on the mesh. In general, a minimum angle condition (shape-regularity) is sufficient in the case of simplicial decompositions, thus allowing for highly adapted meshes. It has also been shown in [AD99] that a maximum angle condition (or an analogous condition if $n = 3$) is sufficient in the case of the lowest-order Raviart–Thomas and Raviart–Thomas–Nédélec spaces.

We also shall require the inf-sup condition

$$\sup_{\vec{q}_h \in \vec{Q}_h} \frac{(\operatorname{div} \vec{q}_h, v_h)}{\|\vec{q}\|_{H(\mathrm{div};\Omega)}} \geq \beta\|v_h\|_{L_2(\Omega)}, \tag{2.11}$$

which may be derived from the commuting diagram property and (2.9); cf. [BF91, Prop. 2.8]. Here $\|\vec{q}\|_{H(\mathrm{div};\Omega)} = \|\vec{q}\|_{[L_2(\Omega)]^n} + \|\operatorname{div} \vec{q}\|_{L_2(\Omega)}$.

These general assumptions are satisfied by all the usual finite element spaces used in the context of mixed methods for elliptic problems. We refer the reader to [BF91, Chap. 3] for an overview of such spaces.

2.4. Description of the *BDM* family of elements. Here we give a brief description of the BDM family of elements for triangular decompositions in two dimensions. The same basic description also holds for simplicial decompositions in three dimensions. We follow [BDM85] and [BF91, Chap. 3].

We assume that τ_h is a triangulation of Ω, and we now require that each triangle $T \in \tau_h$ has straight edges. We note that one may define the BDM elements so that boundary triangles are allowed to have one curved edge, but our computational examples do not involve elements with curved edges and, for brevity, we thus restrict our definition to elements with straight edges. For any given triangle T, we denote the edges by e_i, $i = 1, 2, 3$, and denote the normal vector to the ith edge by $\vec{n}_i$. For a fixed integer $k \geq 1$, we define

$$\vec{Q}_h = \{\vec{q}_h \in H(\mathrm{div};\Omega) \text{ s.t. } \vec{q}_h|_T \in [P_k(T)]^n\},$$
$$V_h = \{v_h \in L_2(\Omega) \text{ s.t. } v_h|_T \in P_{k-1}(T)\},$$

where P_k denotes all polynomials having degree less than or equal to k. Note that the condition that $\vec{q}_h \in H(\text{div};\Omega)$ is equivalent to requiring continuity of the normal component $\vec{q}_h \cdot \vec{n}_i$ across element interfaces, and note that there are no continuity requirements placed on members of V_h across element interfaces.

The L_2-projection P_h is defined by

$$(u - P_h u, v_h) = 0, \quad v_h \in V_h.$$

We next define the vector projection operator Π_h. We first define the edge space $R_k(\partial T)$ by

$$R_k(\partial T) = \{\phi \in L_2(\partial T) \text{ s.t. } \phi|_{e_i} \in P_k(e_i), \quad i = 1, 2, 3\}.$$

Then for $\vec{q} \in W$, we define $\Pi_h \vec{q}$ by

$$\begin{aligned} \langle(\vec{q} - \Pi_h\vec{q}) \cdot \vec{n}, p_k\rangle &= 0, \quad p_k \in R_k(\partial T), \\ (\vec{q} - \Pi_h\vec{q}, \nabla p_{k-1}) &= 0, \quad p_{k-1} \in P_{k-1}(T), \\ (\vec{q} - \Pi_h\vec{q}, \phi_k) &= 0, \quad \phi_k \in \Phi_k, \end{aligned}$$

where $\Phi_k = \{\phi_k \in [P_k(T)]^n \text{ s.t. } \operatorname{div}\phi_k = 0, \quad \phi_k \cdot n|_{\partial T} = 0\}$. The operator Π_h is uniquely defined by these conditions, satisfies the commuting diagram property (2.8), and approximates to order $k+1$ in L_2 when $\vec{q}$ is smooth enough and the mesh satisfies a minimum angle condition. For proofs of these properties, we again refer to [BDM85] and [BF91]. It is also clear that Π_h is the identity when restricted to $\vec{Q}_h$.

3. Proofs of Theorems 1.1, 1.2, and 1.4.

3.1. Proof of Theorem 1.1. For the sake of completeness and comparison, we first prove Theorem 1.1. Using the triangle inequality, we have

$$\|\vec{p} - \vec{p}_h\| \leq \|\vec{p} - \Pi_h\vec{p}\| + \|\Pi_h\vec{p} - \vec{p}_h\|, \tag{3.1}$$

where for the sake of brevity we have abbreviated $\|\cdot\|_{[L_2(\Omega)]^n}$ to $\|\cdot\|$. We use the same abbreviation for $\|\cdot\|_{L_2(\Omega)}$.

Since A is uniformly positive definite with uniformly bounded entries, A^{-1} is also uniformly positive definite with uniformly bounded entries. Thus we have

$$\begin{aligned} \|\Pi_h\vec{p} - \vec{p}_h\|^2 &\leq C(A)(A^{-1}(\Pi_h\vec{p} - \vec{p}_h), \Pi_h\vec{p} - \vec{p}_h) \\ &\leq C(|(A^{-1}(\Pi_h\vec{p} - \vec{p}), \Pi_h\vec{p} - \vec{p}_h)| + |(A^{-1}(\vec{p} - \vec{p}_h), \Pi_h\vec{p} - \vec{p}_h)|) \\ &\leq C(\|\vec{p} - \Pi_h\vec{p}\|\|\Pi_h\vec{p} - \vec{p}_h\| + |(A^{-1}(\vec{p} - \vec{p}_h), \Pi_h\vec{p} - \vec{p}_h)|) \\ &\leq C(\|\vec{p} - \Pi_h\vec{p}\|^2 + |(A^{-1}(\vec{p} - \vec{p}_h), \Pi_h\vec{p} - \vec{p}_h)|) + \frac{1}{2}\|\Pi_h\vec{p} - \vec{p}_h\|^2. \end{aligned} \tag{3.2}$$

Kicking back the last term above and combining (3.2) with (3.1) yields

$$\|\vec{p} - \vec{p}_h\|^2 \leq C\|\vec{p} - \Pi_h\vec{p}\|^2 + C|(A^{-1}(\vec{p} - \vec{p}_h), \Pi_h\vec{p} - \vec{p}_h)|. \tag{3.3}$$

Combining the commuting diagram property (2.8) with the error equation (2.6b), while recalling that $\vec{b} = 0$ and $c = 0$, yields $\operatorname{div}(\Pi_h\vec{p} - \vec{p}_h) = 0$. Then noting that $\operatorname{div}(\Pi_h\vec{p} - \vec{p}_h) \in V_h$ and using (2.6a), we obtain

$$(A^{-1}(\vec{p} - \vec{p}_h), \Pi_h\vec{p} - \vec{p}_h) = (\operatorname{div}(\Pi_h\vec{p} - \vec{p}_h), u - u_h) = 0.$$

Combining the last equation with (3.3) yields the desired result. $\square$

3.2. Proof of Theorem 1.2. In this section we prove the estimate (1.7). Existence and uniqueness are proven in section 3.4. The proof of the error estimate (1.7) from Theorem 1.2 is the same as that of Theorem 1.1 through (3.3); that is, we have

$$\|\vec{p}-\vec{p}_h\|^2 \le C\|\vec{p}-\Pi_h\vec{p}\|^2 + C|(A^{-1}(\vec{p}-\vec{p}_h), \Pi_h\vec{p}-\vec{p}_h)|. \tag{3.4}$$

Using the error equation (2.6a), we obtain

$$\begin{aligned}(A^{-1}(\vec{p}-\vec{p}_h), \Pi_h\vec{p}-\vec{p}_h) &= (\operatorname{div}(\Pi_h\vec{p}-\vec{p}_h), u-u_h)\\ &= (\operatorname{div}(\Pi_h\vec{p}-\vec{p}_h), P_hu-u_h).\end{aligned} \tag{3.5}$$

We next proceed with a duality argument. Let ϕ be the weak solution to the boundary value problem

$$\begin{aligned}-\operatorname{div}(A^*\nabla\phi+\vec{b}\phi)+c\phi &= \operatorname{div}(\Pi_h\vec{p}-\vec{p}_h) \text{ in } \Omega,\\ \phi &= 0 \text{ on } \partial\Omega.\end{aligned}$$

Letting $\vec{r} = -(A^*\nabla\phi+\vec{b}\phi)$, we may rewrite the above in the mixed form

$$(A^{-*}\vec{r}, \vec{q}) - (\operatorname{div}\vec{q}, \phi) + (A^{-*}\vec{b}\phi, \vec{q}) = 0, \tag{3.6a}$$

$$(\operatorname{div}\vec{r}, v) + (c\phi, v) = (\operatorname{div}(\Pi_h\vec{p}-\vec{p}_h), v) \tag{3.6b}$$

for $\{\vec{q}, v\} \in H(\operatorname{div};\Omega)\times L_2(\Omega)$. Combining (3.5) and (3.6b) yields

$$(A^{-1}(\vec{p}-\vec{p}_h), \Pi_h\vec{p}-\vec{p}_h) = (\operatorname{div}\vec{r}, P_hu-u_h) + (c\phi, P_hu-u_h) \equiv I + II. \tag{3.7}$$

Employing the commuting diagram property (2.8) and the error equation (2.6a) yields

$$\begin{aligned}I = (\operatorname{div}\vec{r}, P_hu-u_h) &= (P_h\operatorname{div}\vec{r}, P_hu-u_h) = (P_h\operatorname{div}\vec{r}, u-u_h)\\ &= (\operatorname{div}\Pi_h\vec{r}, u-u_h) = (A^{-1}(\vec{p}-\vec{p}_h), \Pi_h\vec{r})\\ &= (A^{-1}(\vec{p}-\vec{p}_h), \Pi_h\vec{r}-\vec{r}) + (A^{-1}(\vec{p}-\vec{p}_h), \vec{r}).\end{aligned} \tag{3.8}$$

Using (3.6a), we have

$$(A^{-1}(\vec{p}-\vec{p}_h), \vec{r}) = (A^{-*}\vec{r}, \vec{p}-\vec{p}_h) = (\operatorname{div}(\vec{p}-\vec{p}_h), \phi) - (A^{-*}\vec{b}\phi, \vec{p}-\vec{p}_h). \tag{3.9}$$

We next use the commuting diagram property (2.8), the error equation (2.6b), and the dual mixed equation (3.6a) to obtain

$$\begin{aligned}(\operatorname{div}(\vec{p}-\vec{p}_h), \phi) &= (\operatorname{div}(\vec{p}-\vec{p}_h), \phi-P_h\phi) + (\operatorname{div}(\vec{p}-\vec{p}_h), P_h\phi)\\ &= (\operatorname{div}(\vec{p}-\Pi_h\vec{p}), \phi-P_h\phi) - (c(u-u_h), P_h\phi) + (\vec{b}.A^{-1}(\vec{p}-\vec{p}_h), P_h\phi)\\ &= (\operatorname{div}(\vec{p}-\Pi_h\vec{p}), \phi) - (c(u-u_h), P_h\phi) + (\vec{b}.A^{-1}(\vec{p}-\vec{p}_h), P_h\phi)\\ &= (A^{-*}\vec{r}+A^{-*}\vec{b}\phi, \vec{p}-\Pi_h\vec{p}) - (c(u-u_h), P_h\phi) + (\vec{b}.A^{-1}(\vec{p}-\vec{p}_h), P_h\phi).\end{aligned} \tag{3.10}$$

Combining (3.9) and (3.10) yields

$$\begin{aligned}&(A^{-1}(\vec{p}-\vec{p}_h), \vec{r})\\ &= (A^{-*}\vec{r}+A^{-*}\vec{b}\phi, \vec{p}-\Pi_h\vec{p}) - (c(u-u_h), P_h\phi) + (\vec{p}-\vec{p}_h, A^{-*}\vec{b}(P_h\phi-\phi)).\end{aligned} \tag{3.11}$$

We then combine (3.11) and (3.8) to obtain

$$I = (A^{-1}(\vec{p} - \vec{p}_h), \Pi_h \vec{r} - \vec{r}) + (A^{-*}\vec{r} + A^{-*}\vec{b}\phi, p - \Pi_h \vec{p})$$
$$-(c(u - u_h), P_h\phi) + (\vec{p} - \vec{p}_h, A^{-*}\vec{b}(P_h\phi - \phi)),$$

and, after combining with (3.7) and rearranging terms, we have

$$\begin{aligned} I + II &= [(A^{-1}(\vec{p} - \vec{p}_h), \Pi_h \vec{r} - \vec{r}) + (\vec{p} - \vec{p}_h, A^{-*}\vec{b}(P_h\phi - \phi))] \\ &+[(A^{-*}\vec{r} + A^{-*}\vec{b}\phi, \vec{p} - \Pi_h \vec{p})] + [(P_h u - u_h, c\phi) - (c(u - u_h), P_h\phi)] \\ &\equiv III + IV + V. \end{aligned} \tag{3.12}$$

We now estimate the terms above. Using the H^2 regularity of the dual problem and the approximation assumptions (2.9) and (2.10) yields

$$\begin{aligned} III &= (\vec{p} - \vec{p}_h, A^{-*}(\Pi_h \vec{r} - \vec{r})) + (\vec{p} - \vec{p}_h, A^{-*}\vec{b}(P_h\phi - \phi)) \\ &\le C\|\vec{p} - \vec{p}_h\|(\|\vec{r} - \Pi_h \vec{r}\| + \|\phi - P_h\phi\|) \\ &\le Ch\|\vec{p} - \vec{p}_h\|\|\phi\|_{H^2(\Omega)} \le Ch\|\vec{p} - \vec{p}_h\|\|\operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)\|. \end{aligned}$$

Employing the commuting diagram property (2.8) and the error equation (2.6b), we have

$$\begin{aligned} \|\operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)\|^2 &= (\operatorname{div}(\Pi_h \vec{p} - \vec{p}_h), \operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)) \\ &= (\operatorname{div}(\vec{p} - \vec{p}_h), \operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)) \\ &= (\vec{b}.A^{-1}(\vec{p} - \vec{p}_h), \operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)) - (c(u - u_h), \operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)) \\ &\le C(\|\vec{p} - \vec{p}_h\| + \|u - u_h\|)\|\operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)\|. \end{aligned}$$

Thus

$$\|\operatorname{div}(\Pi_h \vec{p} - \vec{p}_h)\| \le C(\|\vec{p} - \vec{p}_h\| + \|u - u_h\|),$$

and for any $\epsilon_1 > 0$,

$$\begin{aligned} III &\le Ch\|\vec{p} - \vec{p}_h\|(\|\vec{p} - \vec{p}_h\| + \|u - u_h\|) \\ &\le Ch\|\vec{p} - \vec{p}_h\|^2 + C\epsilon_1\|\vec{p} - \vec{p}_h\|^2 + \frac{Ch^2}{\epsilon_1}\|u - u_h\|^2. \end{aligned}$$

We next employ the energy-type regularity assumption (2.5) to obtain

$$\begin{aligned} IV &= (A^{-*}\vec{r} + A^{-*}\vec{b}\phi, \vec{p} - \Pi_h \vec{p}) \le C\|\vec{p} - \Pi_h \vec{p}\|\|\phi\|_{H^1(\Omega)} \\ &\le C\|\vec{p} - \Pi_h \vec{p}\|\|\Pi_h \vec{p} - \vec{p}_h\| \le C\|\vec{p} - \Pi_h \vec{p}\|(\|\Pi_h \vec{p} - \vec{p}\| + \|\vec{p} - \vec{p}_h\|) \\ &\le C\|\vec{p} - \Pi_h \vec{p}\|^2 + \frac{C}{\epsilon_2}\|\vec{p} - \Pi_h \vec{p}\|^2 + C\epsilon_2\|\vec{p} - \vec{p}_h\|^2 \end{aligned}$$

for any $\epsilon_2 > 0$.

Next we compute

$$\begin{aligned} V &= (P_h u - u_h, c\phi) - (u - u_h, cP_h\phi) \\ &= (u - u_h, P_h(c\phi)) - (u - u_h, cP_h\phi) \\ &= (u - u_h, P_h(c\phi) - c\phi) + (u - u_h, c\phi) - (u - u_h, cP_h\phi) \\ &= (u - u_h, P_h(c\phi) - c\phi) + (u - u_h, c(\phi - P_h\phi)). \end{aligned} \tag{3.13}$$

Again employing the energy-type regularity assumption (2.5) yields

$$\begin{aligned}(u - u_h, P_h(c\phi) - c\phi) + (u - u_h, c\phi - cP_h\phi)\\ \le \|u - u_h\|(\|P_h(c\phi) - c\phi\| + \|c\phi - cP_h\phi\|)\\ \le C(c)h\|u - u_h\|\|\phi\|_{H^1(\Omega)} \le C(c)h\|u - u_h\|\|\Pi_h\vec{p} - \vec{p}_h\|.\end{aligned}$$

We thus obtain, for any $\epsilon_3 > 0$,

$$\begin{aligned}V &\le Ch\|u - u_h\|(\|\vec{p} - \vec{p}_h\| + \|\vec{p} - \Pi_h\vec{p}\|)\\ &\le C\left(h^2\left(1 + \frac{1}{\epsilon_3}\right)\|u - u_h\|^2 + \epsilon_3\|\vec{p} - \vec{p}_h\|^2 + \|\vec{p} - \Pi_h\vec{p}\|^2\right).\end{aligned}$$

We finally collect terms III, IV, and V into (3.12), which we substitute back into (3.7) and finally (3.4) to obtain

$$\|\vec{p} - \vec{p}_h\|^2 \le C_1(\|\vec{p} - \Pi_h\vec{p}\|^2 + h^2\|u - u_h\|^2) + C_2(h + \epsilon_1 + \epsilon_2 + \epsilon_3)\|\vec{p} - \vec{p}_h\|^2,$$

where C_1 depends on the ϵ_i's but C_2 does not. Taking h and the ϵ_i's small enough to kick back the last term above yields

$$\|\vec{p} - \vec{p}_h\| \le C(\|\vec{p} - \Pi_h\vec{p}\| + h\|u - u_h\|). \tag{3.14}$$

The proof of estimate (1.7) from Theorem 1.2 is completed using the following lemma and a further kick-back argument for h small enough.

LEMMA 3.1. *Assume that all assumptions of Theorem* 1.2 *are satisfied and h is taken to be small enough. Then*

$$\|u - u_h\|_{L_2(\Omega)} \le C(\|u - P_hu\|_{L_2(\Omega)} + \|\vec{p} - \vec{p}_h\|_{[L_2(\Omega)]^n}). \tag{3.15}$$

Proof. Clearly it is sufficient to bound $\|P_hu - u_h\|$ by the right-hand side of (3.15). Using the inf-sup condition (2.11), the commuting diagram property (2.8), and the error equation (2.6a), we find

$$\begin{aligned}\beta\|P_hu - u_h\| &\le \sup_{\vec{q}_h\in\vec{Q}_h} \frac{(\operatorname{div}\vec{q}_h, P_hu - u_h)}{\|\vec{q}_h\|_{H(\operatorname{div};\Omega)}} = \sup_{\vec{q}_h\in\vec{Q}_h} \frac{(\operatorname{div}\vec{q}_h, u - u_h)}{\|\vec{q}_h\|_{H(\operatorname{div};\Omega)}}\\ &= \sup_{\vec{q}_h\in\vec{Q}_h} \frac{(A^{-1}(\vec{p} - \vec{p}_h), \vec{q}_h)}{\|\vec{q}_h\|_{H(\operatorname{div};\Omega)}} \le C\|\vec{p} - \vec{p}_h\|_{[L_2(\Omega)]^n}.\end{aligned}$$

This completes the proof of Lemma 3.1 and thus of (1.7). ∎

It is possible to prove an estimate similar to (3.15), which is sufficient to complete the proof of (1.7), by using a duality argument similar to that employed in the proof of Corollary 3.4 instead of using the inf-sup condition.

3.3. Proof of Theorem 1.4. Focusing our attention first on the error estimate (1.8), we again delay the proof of existence and uniqueness until section 3.4. As in the proof of Theorem 1.1, we have

$$\|\tilde{p} - \tilde{p}_h\|^2 \le C\|\tilde{p} - \Pi_h\tilde{p}\|^2 + C|(A^{-1}(\tilde{p} - \tilde{p}_h), \Pi_h\tilde{p} - \tilde{p}_h)|. \tag{3.16}$$

From (2.7a), we have

$$\begin{aligned}&(A^{-1}(\tilde{p} - \tilde{p}_h), \Pi_h\tilde{p} - \tilde{p}_h)\\ &= (\operatorname{div}(\Pi_h\tilde{p} - \tilde{p}_h), u - u_h) - (A^{-1}\vec{b}(u - u_h), \Pi_h\tilde{p} - \tilde{p}_h)\\ &\le C\|\operatorname{div}(\Pi_h\tilde{p} - \tilde{p}_h)\|\|u - u_h\| + \|u - u_h\|\|\Pi_h\tilde{p} - \tilde{p}_h\|\\ &\le C\|u - u_h\|^2 + \|\operatorname{div}(\Pi_h\tilde{p} - \tilde{p}_h)\|^2 + \epsilon\|\Pi_h\tilde{p} - \tilde{p}_h\|^2. \qquad (3.17)\end{aligned}$$

Noting that $\epsilon\|\Pi_h\tilde{p}-\tilde{p}_h\|^2 \le 2\epsilon(\|\tilde{p}-\Pi_h\tilde{p}\|^2+\|\tilde{p}-\tilde{p}_h\|^2)$, we combine (3.16) with (3.17) and take ϵ small enough to obtain

$$\|\tilde{p}-\tilde{p}_h\|^2 \le C(\|\tilde{p}-\Pi_h\tilde{p}\|^2 + \|\operatorname{div}(\Pi_h\tilde{p}-\tilde{p}_h)\|^2 + \|u-u_h\|^2). \tag{3.18}$$

From the commuting diagram property (2.8) and the error equation (2.7b), we deduce

$$\begin{aligned}&(\operatorname{div}(\Pi_h\tilde{p}-\tilde{p}_h), \operatorname{div}(\Pi_h\tilde{p}-\tilde{p}_h)) = (\operatorname{div}(\tilde{p}-\tilde{p}_h), \operatorname{div}(\Pi_h\tilde{p}-\tilde{p}_h))\\ &= -(c(u-u_h), \operatorname{div}(\Pi_h\tilde{p}-\tilde{p}_h)) \le C\|u-u_h\|\|\operatorname{div}(\Pi_h\tilde{p}-\tilde{p}_h)\|.\end{aligned}$$

Substituting $\|\operatorname{div}(\Pi_h\tilde{p}-\tilde{p}_h)\| \le C\|u-u_h\|$ into (3.18), we obtain

$$\|\tilde{p}-\tilde{p}_h\| \le C(\|\tilde{p}-\Pi_h\tilde{p}\| + \|u-u_h\|).$$

Lemma 3.2 and a kick-back argument for h small enough complete the proof of the estimate (1.8) from Theorem 1.4.

LEMMA 3.2. *Assume that all the assumptions of Theorem* 1.4 *are satisfied and that h is small enough. Then*

$$\|u-u_h\|_{L_2(\Omega)} \le C(\|u-P_hu\|_{L_2(\Omega)} + \|\tilde{p}-\Pi_h\tilde{p}\|_{[L_2(\Omega)]^n} + h\|\tilde{p}-\tilde{p}_h\|_{[L_2(\Omega)]^n}). \tag{3.19}$$

Proof. Clearly it is sufficient to bound $\|P_hu-u_h\|$ by the right-hand side of (3.19). Let ϕ be the weak solution to

$$\begin{aligned}-\operatorname{div}(A^*\nabla\phi) + \vec{b}.\nabla\phi + c\phi = P_hu-u_h \text{ in } \Omega,\\ \phi = 0 \text{ on } \partial\Omega.\end{aligned}$$

With $\vec{r} = -A^*\nabla\phi$, we write the above problem in the mixed form

$$(A^{-*}\vec{r},\vec{q}) - (\operatorname{div}\vec{q},\phi) = 0, \tag{3.20a}$$

$$(\operatorname{div}\vec{r}, v) - (\vec{b}.A^{-*}\vec{r}, v) + (c\phi, v) = (P_hu-u_h, v) \tag{3.20b}$$

for $\{\vec{q}, v\} \in H(\operatorname{div};\Omega)\times L_2(\Omega)$. Then

$$\begin{aligned}&(P_hu-u_h, P_hu-u_h)\\ &= (\operatorname{div}\vec{r}, P_hu-u_h) + (-\vec{b}.A^{-*}\vec{r}+c\phi, P_hu-u_h) \equiv I + II.\end{aligned} \tag{3.21}$$

Using the commuting diagram property (2.8) and the error equation (2.7a), we have

$$\begin{aligned}I &= (\operatorname{div}\vec{r}, P_hu-u_h) = (P_h\operatorname{div}\vec{r}, P_hu-u_h) = (\operatorname{div}\Pi_h\vec{r}, u-u_h)\\ &= (A^{-1}(\tilde{p}-\tilde{p}_h), \Pi_h\vec{r}) + (A^{-1}\vec{b}(u-u_h), \Pi_h\vec{r}) = (A^{-1}(\tilde{p}-\tilde{p}_h), \Pi_h\vec{r}-\vec{r})\\ &\quad + (A^{-1}(\tilde{p}-\tilde{p}_h), \vec{r}) + (A^{-1}\vec{b}(u-u_h), \Pi_h\vec{r}-\vec{r}) + (A^{-1}\vec{b}(u-u_h), \vec{r}).\end{aligned}$$

We next use the dual mixed equation (3.20a), the commuting diagram property (2.8), the error equation (2.7b), and then again use the dual mixed equation (3.20a) to calculate

$$\begin{aligned}(A^{-1}(\tilde{p}-\tilde{p}_h), \vec{r}) &= (A^{-*}\vec{r}, \tilde{p}-\tilde{p}_h) = (\operatorname{div}(\tilde{p}-\tilde{p}_h), \phi)\\ &= (\operatorname{div}(\tilde{p}-\tilde{p}_h), \phi-P_h\phi) + (\operatorname{div}(\tilde{p}-\tilde{p}_h), P_h\phi)\\ &= (\operatorname{div}(\tilde{p}-\Pi_h\tilde{p}), \phi-P_h\phi) + (\operatorname{div}(\tilde{p}-\tilde{p}_h), P_h\phi)\\ &= (\operatorname{div}(\tilde{p}-\Pi_h\tilde{p}), \phi) - (u-u_h, cP_h\phi) = (A^{-1}(\tilde{p}-\Pi_h\tilde{p}), \vec{r}) - (u-u_h, cP_h\phi).\end{aligned}$$

Combining the last three equations and rearranging yields

$$\begin{aligned}
I + II &= (A^{-1}(\tilde{p} - \tilde{p}_h), \Pi_h \vec{r} - \vec{r}) + (A^{-1}(\tilde{p} - \Pi_h \tilde{p}), \vec{r}) - (u - u_h, cP_h\phi) \\
&\quad + (A^{-1}\vec{b}(u - u_h), \Pi_h \vec{r} - \vec{r}) + (A^{-1}\vec{b}(u - u_h), \vec{r}) + (-\vec{b}.A^{-*}\vec{r} + c\phi, P_h u - u_h) \\
&= [(A^{-1}(\tilde{p} - \tilde{p}_h + \vec{b}(u - u_h)), \Pi_h \vec{r} - \vec{r}) + (A^{-1}(\tilde{p} - \Pi_h \tilde{p}), \vec{r}) \\
&\quad + (A^{-1}\vec{b}(u - P_h u), \vec{r})] + [(P_h u - u_h, c\phi) - (u - u_h, cP_h\phi)] \equiv III + IV. \qquad (3.22)
\end{aligned}$$

Using the definition of $\vec{r}$, the approximation properties (2.9) and (2.10) and H^2 regularity yields

$$\begin{aligned}
III &= (A^{-1}(\tilde{p} - \tilde{p}_h), \Pi_h \vec{r} - \vec{r}) + (A^{-1}\vec{b}(u - u_h), \Pi_h \vec{r} - \vec{r}) \\
&\quad + (A^{-1}(\tilde{p} - \Pi_h \tilde{p}), \vec{r}) + (A^{-1}\vec{b}(u - P_h u), \vec{r}) \\
&\le C(h\|\tilde{p} - \tilde{p}_h\| + h\|u - u_h\| + \|\tilde{p} - \Pi_h \tilde{p}\| + \|u - P_h u\|)\|\phi\|_{H^2(\Omega)} \\
&\le C(h\|\tilde{p} - \tilde{p}_h\| + h\|P_h u - u_h\| + \|\tilde{p} - \Pi_h \tilde{p}\| + \|u - P_h u\|)\|P_h u - u_h\|.
\end{aligned}$$

We similarly compute

$$\begin{aligned}
IV &= (P_h u - u_h, c\phi) - (u - u_h, cP_h\phi) \\
&= (P_h u - u_h, c\phi - cP_h\phi) + (P_h u - u_h, cP_h\phi) - (u - u_h, cP_h\phi) \\
&= (P_h u - u_h, c(\phi - P_h\phi)) + (P_h u - u, cP_h\phi) \\
&\le C(h\|P_h u - u_h\|^2 + \|u - P_h u\|\|P_h u - u_h\|).
\end{aligned}$$

We next combine the last two inequalities with (3.21) and (3.22) to obtain

$$\begin{aligned}
&\|P_h u - u_h\|^2 \\
&\le C(h\|\tilde{p} - \tilde{p}_h\| + h\|P_h u - u_h\| + \|u - P_h u\| + \|\tilde{p} - \Pi_h \tilde{p}\|)\|P_h u - u_h\|. \qquad (3.23)
\end{aligned}$$

In order to complete the proof we divide the above inequality by $\|P_h u - u_h\|$, then take h small enough to kick back the second term in parentheses. ∎

Remark 3.3. We note here that the proof of Lemma 3.2 requires a duality argument, whereas the proof of the analogous lemma in the divergence-form case, Lemma 3.1, requires only a simple argument employing the inf-sup condition.

3.4. Existence, uniqueness, and estimates for $u - u_h$. Here we prove existence and uniqueness of solutions to problems (1.3) and (1.4) as stated in Theorems 1.2 and 1.4. We also give estimates for the scalar error $u - u_h$.

We first obtain estimates for $\|u - u_h\|$. Here, instead of (2.10), we assume that for some $k \ge 1$ and for all $v \in H^m(\Omega)$,

$$\|v - P_h v\| \le Ch^m \|v\|_{H^m(\Omega)} \text{ for } 1 \le m \le k. \qquad (3.24)$$

We recall the Kronecker delta δ_{ij}, where $\delta_{ij} = 0$ if $i \ne j$ and $\delta_{ij} = 1$ if $i = j$.

Corollary 3.4. *Let* $\{u - u_h, p - p_h\}$ *satisfy error equations* (2.6a) *and* (2.6b) *or* (2.7a) *and* (2.7b), *assume that* (3.24) *holds, and let the corresponding assumptions for Theorem* 1.2 *or Theorem* 1.4 *hold. Then for h small enough,*

$$\begin{aligned}
\|u - u_h\|_{L_2(\Omega)} &\le C(\|u - P_h u\|_{L_2(\Omega)} \\
&\quad + h\|p - \Pi_h p\|_{[L_2(\Omega)]^n} + h^{2-\delta_{1k}}\|\operatorname{div} p - P_h \operatorname{div} p\|_{L_2(\Omega)}). \qquad (3.25)
\end{aligned}$$

Proof. We first prove the corollary for the divergence-form case. Clearly, it is sufficient to bound $\|P_h u - u_h\|$ by the right-hand side of (3.25). Let ϕ be the weak solution to

$$-\operatorname{div}(A^* \nabla \phi + \vec{b}\phi) + c\phi = P_h u - u_h \text{ in } \Omega,$$
$$\phi = 0 \text{ on } \partial\Omega.$$

With $\vec{r} = -(A^* \nabla \phi + \vec{b}\phi)$, as in the duality argument used in the proof of Theorem 1.2, we rewrite the above in the mixed form

$$(A^{-*}\vec{r}, \vec{q}) - (\operatorname{div} \vec{q}, \phi) + (A^{-*}\vec{b}\phi, \vec{q}) = 0, \tag{3.26a}$$
$$(\operatorname{div} \vec{r}, v) + (c\phi, v) = (P_h u - u_h, v) \tag{3.26b}$$

for $\{\vec{q}, v\} \in H(\operatorname{div}; \Omega) \times L_2(\Omega)$. We then may write $(P_h u - u_h, P_h u - u_h) = (\operatorname{div} \vec{r} + c\phi, P_h u - u_h)$. We may manipulate this equation exactly as in (3.7)–(3.12) and then manipulate the term V in (3.12) as in (3.13) to obtain

$$\begin{aligned}\|P_h u - u_h\|^2 &= (P_h u - u_h, P_h u - u_h) \\ &= (A^{-1}(\vec{p} - \vec{p}_h), \Pi_h \vec{r} - \vec{r}) + (\vec{p} - \vec{p}_h, A^{-*}\vec{b}(P_h \phi - \phi)) \\ &\quad + (A^{-*}\vec{r} + A^{-*}\vec{b}\phi, \vec{p} - \Pi_h \vec{p}) + (u - u_h, P_h(c\phi) - c\phi) + (u - u_h, c(\phi - P_h \phi)).\end{aligned} \tag{3.27}$$

We next use the mixed equation (3.26a) and the commuting diagram property (2.8) to find

$$(A^{-*}\vec{r} + A^{-*}\vec{b}\phi, \vec{p} - \Pi_h \vec{p}) = (\operatorname{div}(\vec{p} - \Pi_h \vec{p}), \phi) = (\operatorname{div} \vec{p} - P_h \operatorname{div} \vec{p}, \phi - P_h \phi). \tag{3.28}$$

Inserting (3.28) into (3.27) and using the approximation assumptions (2.9) and (3.24) and H^2 regularity, we thus obtain

$$\begin{aligned}\|P_h u - u_h\|^2 &\le C(h\|\vec{p} - \vec{p}_h\| + h^{2-\delta_{1k}} \|\operatorname{div} \vec{p} - P_h \operatorname{div} \vec{p}\| + h\|u - u_h\|)\|\phi\|_{H^2(\Omega)} \\ &\le C(h\|\vec{p} - \vec{p}_h\| + h^{2-\delta_{1k}} \|\operatorname{div} \vec{p} - P_h \operatorname{div} \vec{p}\| \\ &\qquad + h\|u - P_h u\| + h\|P_h u - u_h\|)\|P_h u - u_h\|.\end{aligned}$$

Dividing by $\|P_h u - u_h\|$ yields

$$\begin{aligned}\|P_h u - u_h\| \le C(&h\|u - P_h u\| + h\|\vec{p} - \vec{p}_h\| \\ &+ h^{2-\delta_{1k}} \|\operatorname{div} \vec{p} - P_h \operatorname{div} \vec{p}\| + h\|P_h u - u_h\|).\end{aligned} \tag{3.29}$$

To complete the proof, we insert estimate (1.7) into (3.29) and take h small enough to kick back the last term above.

The estimate for the conservation-form case may be obtained in similar fashion by treating the term $(A^{-1}(\tilde{p} - \Pi_h \tilde{p}), \vec{r})$ in (3.22) as in (3.28) and then proceeding as above. ∎

Remark 3.5. One may trivially obtain an estimate of the form

$$\|u - u_h\| \le C(\|u - P_h u\| + \|p - \Pi_h p\|)$$

by combining (1.7) with (3.15) or (1.8) with (3.19), but such an estimate would require H^{k+1} regularity in order to produce an $O(h^k)$ error. Except in the lowest-order case $k = 1$, the estimate (3.25) requires only H^k regularity to produce an $O(h^k)$ error.

Existence and uniqueness for methods (1.3) and (1.4) follow easily from the estimates given in Theorem 1.2, Theorem 1.4, and Corollary 3.4. Our argument is similar to that of [Sch74] in that we derive a priori error estimates, without first proving existence and uniqueness, and we then use the a priori estimates to prove uniqueness. We note that if either (1.3) or (1.4) has two solutions $\{u_h, p_h\}$ and $\{\hat{u}_h, \hat{p}_h\}$ in $\vec{Q}_h \times V_h$, then the pair $\{u_h - \hat{u}_h, p_h - \hat{p}_h\}$ satisfies the corresponding error equations (2.6a) and (2.6b) or (2.7a) and (2.7b), respectively. Taking h small enough allows us to apply Theorem 1.2 or Theorem 1.4 along with Corollary 3.4 (note that how small h is required to be is dependent on the coefficients A, $\vec{b}$, and c and the domain Ω, but not on u, p, etc.). Since $\{p_h, u_h\}$ is in the mixed approximating space and our projection operators are the identity on this space by assumption, we then have $\|u_h - \hat{u}_h\| = \|p_h - \hat{p}_h\| = 0$. Thus uniqueness holds, and since (1.3) and (1.4) are finite-dimensional systems of linear equations, existence holds as well. The proofs of Theorems 1.2 and 1.4 are thus completed. □

4. Computational results. In this section we provide computational evidence that Theorem 1.4 for conservation-form mixed methods is sharp when the BDM_k spaces are used. In particular, the vector variable $\tilde{p}_h$ converges to $\tilde{p}$ with suboptimal order in the conservation-form mixed method (1.4).

In our computations, we let Ω be the unit square $(0,1)\times(0,1) \subset \mathbb{R}^2$ and considered the simple model problem

$$\begin{aligned} -\operatorname{div}(\nabla u + \vec{b}u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{4.1}$$

where $\vec{b} = (.9, .9)$ and f was defined by $\vec{b}$ and $u(x,y) = x(x-1)y(y-1)$. Thus in (1.4), A was the 2×2 identity matrix and $c \equiv 0$. The bilinear form associated with (4.1) with the given $\vec{b}$ is easily shown to be coercive over $H_0^1(\Omega)$, and Ω is convex, so uniqueness and regularity of the problem and its dual follow immediately. A simple uniform triangular mesh was employed, and the error $\|\vec{p} - \vec{p}_h\|$ was computed for $h = \frac{1}{2}^j$, where $j = 2, \ldots, 7$ in the case $k = 1$ and $j = 1, \ldots, 6$ in the case $k = 2$. For each pair of results corresponding to a pair $(h, h/2)$ of mesh sizes, an observed rate of convergence was calculated using the formula

$$r_h = \log_2\left(\frac{\|\tilde{p} - \tilde{p}_h\|}{\|\tilde{p} - \tilde{p}_{h/2}\|}\right).$$

Quadrature rules were obtained from [SF73, p. 184]. For experiments involving BDM_1, a 7-point quadrature rule which integrates exactly polynomials of up to degree 5 was used. For BDM_2 experiments, a 13-point rule integrating exactly polynomials of up to degree 7 was used. Thus all integrals in the finite element calculations themselves were exact, and in computing the error integrals the order of the error introduced by numerical quadrature was higher than the order of approximation given by the element spaces and was thus insignificant. A standard interelement Lagrange multiplier scheme for mixed methods (as described in [BDM85], for example) was employed to enable the use of an iterative matrix solver. Since the problem at hand is not symmetric, a conjugate gradient routine could not be used to solve the system of equations obtained from the multiplier method, and a quasi-minimal residual algorithm was used instead with a maximum relative error of about 3×10^{-10} being achieved. Results are displayed in Table 4.1.

TABLE 4.1
Conservation form.

	BDM_1		BDM_2	
j	$\|\tilde{p} - \tilde{p}_h\|_{L_2}$	r_h	$\|\tilde{p} - \tilde{p}_h\|_{L_2}$	r_h
1			.00263	2.151
2	.00688	1.210	.000593	2.075
3	.00298	1.067	.000141	2.028
4	.00142	1.018	.0000345	2.010
5	.000702	1.005	.00000856	2.004
6	.000350	1.001	.00000214	
7	.000175			

TABLE 4.2
Divergence form.

	BDM_1		BDM_2	
n	$\|\vec{p} - \vec{p}_h\|_{L_2}$	r_h	$\|\vec{p} - \vec{p}_h\|_{L_2}$	r_h
1			.00141	2.851
2	.00412	1.975	.000195	2.924
3	.00105	1.991	.0000257	2.966
4	.000264	1.997	.00000329	2.983
5	.0000661	1.999	.00000042	2.992
6	.0000165	1.999	.00000005	
7	.0000042			

We recall that BDM_1 can approximate $\tilde{p}$ to order h^2, and BDM_2 can approximate $\tilde{p}$ to order h^3. Thus the conservation-form method yields a suboptimal order of convergence when the BDM_k elements are used.

For comparison, similar computations were performed for two different divergence-form problems. Table 4.2 gives results for (1.3) when $u = x(x-1)y(y-1)$, as above, and when $\vec{b} = (.9, .9)$ (in the case $k = 1$) or $\vec{b} = (0, 0)$ (in the case $k = 2$). As our theory predicts, optimal rates of convergence are observed.

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REFERENCES

[AD99] G. ACOSTA AND R. G. DURÁN, *The maximum angle condition for mixed and nonconforming elements: Application to the Stokes equation*, SIAM J. Numer. Anal., 37 (1999), pp. 18–36.

[BDM85] F. BREZZI, J. DOUGLAS, JR., AND L. MARINI, *Two families of mixed finite elements for second order elliptic problems*, Numer. Math., 47 (1985), pp. 217–235.

[BF91] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*, Springer-Verlag, New York, 1991.

[Ch94] Z. CHEN, *BDM mixed methods for a nonlinear elliptic problem*, J. Comput. Appl. Math., 53 (1994), pp. 207–223.

[Ch98a] Z. CHEN, *Expanded mixed finite element methods for linear second-order elliptic problems* I, RAIRO Modél. Math. Anal. Numér., 32 (1998), pp. 479–499.

[Ch98b] Z. CHEN, *Expanded mixed finite element methods for quasilinear second-order elliptic problems* II, RAIRO Modél. Math. Anal. Numér., 32 (1998), pp. 501–520.

[DR82] J. Douglas, Jr. and J. E. Roberts, *Mixed finite element methods for second order elliptic problems*, Mat. Apl. Comput., 1 (1982), pp. 91–103.

[DR85] J. Douglas, Jr. and J. E. Roberts, *Global estimates for mixed methods for second order elliptic equations*, Math. Comp., 44 (1985), pp. 39–52.

[Du88] R. G. Durán, *Error analysis in* $L^p, 1 \leq p \leq \infty$*, for mixed finite element methods for linear and quasi-linear elliptic problems*, RAIRO Modél. Math. Anal. Numér., 22 (1988), pp. 371–387.

[FO80] R. Falk and J. Osborn, *Error estimates for mixed methods*, RAIRO Anal. Numér., 14 (1980), pp. 249–277.

[GN88] L. Gastaldi and R. H. Nochetto, *On* L^∞*-accuracy of mixed finite element methods for second order elliptic problems*, Mat. Apl. Comput., 7 (1988), pp. 13–39.

[Gr85] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston, MA, 1985.

[Sch74] A. H. Schatz, *An observation concerning Ritz-Galerkin methods with indefinite bilinear forms*, Math. Comp., 28 (1974), pp. 959–962.

[SF73] G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*, Prentice-Hall, Englewood Cliffs, NJ, 1973.