

# LOCALIZED POINTWISE A POSTERIORI ERROR ESTIMATES FOR GRADIENTS OF PIECEWISE LINEAR FINITE ELEMENT APPROXIMATIONS TO SECOND-ORDER QUASILINEAR ELLIPTIC PROBLEMS\*

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**Abstract.** Two types of pointwise a posteriori error estimates are presented for gradients of finite element approximations of second-order quasilinear elliptic Dirichlet boundary value problems over convex polyhedral domains  $\Omega$  in space dimension  $n \geq 2$ . We first give a residual estimator which is equivalent to  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)}$  up to higher-order terms. The second type of residual estimator is designed to control  $\nabla(u - u_h)$  locally over any subdomain of  $\Omega$ . It is a novel a posteriori counterpart to the localized or weighted a priori estimates of [Sch98]. This estimator is shown to be equivalent (up to higher-order terms) to the error measured in a weighted global norm which depends on the subdomain of interest. All estimates are proved for general shape-regular meshes which may be highly graded and unstructured. The constants in the estimates depend on the unknown solution  $u$  in the nonlinear case, but in a fashion which places minimal restrictions on the regularity of  $u$ .

**Key words.** finite element methods, quasilinear elliptic problems, a posteriori error estimation, pointwise error analysis

**AMS subject classifications.** 65N30, 65N15

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## 1. Introduction and results.

**1.1. Introduction.** We consider finite element approximations to second-order quasilinear elliptic Dirichlet boundary value problems having the form

$$(1.1) \quad \begin{aligned} -\sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(x, u, \nabla u) + F_0(x, u, \nabla u) &= 0 \text{ in } \Omega, \\ u &= b \text{ on } \partial\Omega. \end{aligned}$$

Here  $\Omega$  is a convex polyhedral domain in  $\mathbb{R}^n$ ,  $n \geq 2$ , and we assume that  $u \in C^{1,\alpha}(\bar{\Omega})$  for some  $0 < \alpha \leq 1$ . The coefficients  $F_i(x, z, p)$  are assumed to be elliptic (although not necessarily uniformly so) and to satisfy minimal smoothness requirements given in detail later. Problems ranging from uniformly elliptic equations to highly nonlinear, nonuniformly elliptic equations take the form (1.1). Examples which may be treated with the techniques presented here include uniformly elliptic linear problems, where  $F_i(x, z, p) = \sum_{j=1}^n a_{ij}(x) p_j$ ,  $1 \leq i \leq n$ , and  $F_0(x, z, p) = \vec{b}(x) \cdot p + c(x)z - f(x)$ ; the prescribed mean curvature equation, where  $F_i(x, z, p) = p_i / \sqrt{1 + |p|^2}$ ,  $1 \leq i \leq n$ , and  $F_0(x, z, p) = -H(x)$ ; and mildly nonlinear equations, where  $F_i(x, z, p) = \sum_{j=1}^n a_{ij}(x, z) p_j$ ,  $1 \leq i \leq n$ , and  $F_0(x, z, p) = -f(x)$ .

In this paper we provide two types of computationally efficient residual-based pointwise a posteriori error estimators for the gradient error  $\nabla(u - u_h)$  in the piecewise linear finite element approximation  $u_h$  to  $u$ . We first give estimators which are

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equivalent up to constants and logarithmic factors to  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)}$ . While most a posteriori error estimates in the literature similarly control global norms of the error, the quantity of interest in many practical calculations is dependent only on the solution in some subset  $D$  of  $\Omega$ . The goal in these cases is to refine the mesh enough globally to ensure that the solution in  $\Omega \setminus D$  does not “pollute” the solution in  $D$  while not overrefining in  $\Omega \setminus D$ . To this end, we prove an a posteriori error estimate for  $\|\nabla(u - u_h)\|_{L^\infty(D)}$  which is a novel a posteriori counterpart to the weighted or localized a priori pointwise estimates proved in [Sch98]. The resulting estimators, which we call localized estimators, bound  $\|\nabla(u - u_h)\|_{L^\infty(D)}$  and are essentially equivalent to a certain weighted global norm of  $\nabla(u - u_h)$ . Both types of estimates are valid on general shape-regular meshes and under reasonable regularity assumptions on coefficients and the solution  $u$ . To our knowledge, these estimates are the first to provide pointwise error control for gradients in either global or local norms on highly graded, unstructured meshes.

The  $W_\infty^1$  estimates we give here are in several senses an extension of the framework for a posteriori analysis of nonlinear problems in integral norms which was proposed in [Ver94]. As in that work, our estimates provide a theoretical basis for a posteriori error estimation and adaptive mesh refinement but also suffer from several drawbacks. The first is that we can prove reliability of our estimators only under the uncomputable condition that  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)}$  is small enough. Second, a priori constants appear in our a posteriori upper bounds. Finally, the estimates presented here suffer from a “spectral gap” between the a posteriori upper and lower bounds when the maximum pointwise ratio of the largest and smallest eigenvalues of the coefficient matrix  $[\frac{\partial}{\partial p_j} F_i(x, u, \nabla u)]$  is large. [FV03] proposes a method which essentially eliminates the first two of these problems in the context of a posteriori error estimation in the energy norm for equations of prescribed mean curvature. The third problem mentioned above, ill conditioning resulting from a spectral gap, seems to be an essential feature of residual-type estimators for linear as well as nonlinear problems; cf. [FV03] and [BV00].

In the present work we focus on presenting a basic theory for problems on polyhedral domains. Two important questions which we do not consider here are the case of smooth boundaries and the treatment of the constants arising in our estimates. The first of these questions is important for many nonlinear problems as even theoretical results are not always available on polygonal domains. However, the proper treatment of finite element approximations involving curved boundaries is somewhat technical even when considering a posteriori energy-norm bounds (cf. [DR98]), and we do not wish to clutter our presentation. Second, the constants in our a posteriori estimates depend on the unknown solution  $u$  in nonlinear problems and even in linear problems may depend on the coefficients in a fashion that will require a local weighting of the residuals. In [De05] we combine further theoretical results with computational experiments in order to investigate this problem.

**1.2. Outline of results.** Before outlining our results, we introduce some notation. First note that we shall restrict most of our presentation to a model problem having the form

$$(1.2) \quad \begin{aligned} - \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(x, \nabla u) &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where  $\Omega$  is convex and polygonal. Most of the examples mentioned in the previous

section are of the form (1.2). Extension to more general coefficients is fairly immediate under appropriate assumptions. In section 5 we sketch the necessary modifications and also provide a brief analysis of problems with nonhomogeneous Dirichlet boundary conditions.

Let  $\mathcal{T}$  be a decomposition of  $\Omega$  (now assumed to be polygonal) into shape-regular simplices. Let also  $h_T = |T|^{1/n}$  for each element  $T \in \mathcal{T}$ , let  $\underline{h} = \min_{T \in \mathcal{T}} h_T$ , and let  $S_h$  be the continuous piecewise linear functions which are 0 on  $\partial\Omega$ . We emphasize that we place no restrictions on the mesh other than shape regularity, so that both highly graded and unstructured meshes are allowed throughout. A logarithmic factor which for technical reasons is different when  $n = 2$  also appears in our estimates, and for convenience we define  $\gamma(2) = 2$ ,  $\gamma(n) = 1$  for  $n > 2$ , and  $\ell_{\underline{h},n} = (\log(1/\underline{h}))^{\gamma(n)}$ .

We next define a first-order maximum norm residual. Let  $S$  be a face shared by two elements  $T_1$  and  $T_2$ , and let  $\vec{n}$  be a unit normal on  $S$  (with arbitrary orientation). For  $v_h \in S_h$ , we then define

$$[v_h]_S(x) = \sum_{i=1}^n [F_i(x, \nabla v_h|_{T_1}) - F_i(x, \nabla v_h|_{T_2})] n_i$$

with  $[v_h]_S(x) = 0$  when  $S \subset \partial\Omega$ . Dropping the subscript  $S$  as it will cause no confusion, we define the residual

$$(1.3) \quad \mathcal{E}_T = h_T \|f + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(\cdot, \nabla u_h)\|_{L_\infty(T)} + \|[u_h]\|_{L_\infty(\partial T)}.$$

Our first result is the global estimate

$$(1.4) \quad \frac{1}{\tilde{C}_1} \max_{T \in \mathcal{T}} (\mathcal{E}_T - R_1(T)) \leq \|\nabla(u - u_h)\|_{L_\infty(\Omega)} \leq C_1 \ell_{\underline{h},n} \max_{T \in \mathcal{T}} \mathcal{E}_T + R_2(\Omega),$$

where  $R_i$ ,  $i \geq 1$ , denotes a higher-order term which will be defined more precisely later and  $C_1$  and  $\tilde{C}_1$  depend on the coefficients  $F_i$ ,  $\|\nabla u\|_{L_\infty(\Omega)}$ , and the Dini continuity of  $\nabla u$  in the nonlinear case. Thus our estimators are equivalent to the actual error up to constants, logarithmic factors, and higher-order terms, that is, they are efficient and reliable. We do not attempt to provide asymptotically exact estimators. Besides having obvious application to control of global norms of gradient errors, our global estimates may also be combined with the results of [Noc95] and [DDP00] to establish a posteriori estimates for  $\|u - u_h\|_{L_\infty(\Omega)}$  for nonlinear problems on convex polygonal and polyhedral domains in two and three space dimensions.

In order to provide local error control, we present novel localized estimators which are inspired by the localized or weighted a priori pointwise estimates proved in [Sch98]. These estimates are valid for smooth linear Neumann problems on globally quasi-uniform meshes of size  $h$ . Defining the weight  $\sigma_{x_0}(y) = \frac{h}{|y-x_0|+h}$ , it was shown that

$$|\nabla(u - u_h)(x_0)| \leq \|\sigma_{x_0} \nabla(u - u_h)\|_{L_\infty(\Omega)} \leq C \min_{\chi \in S_h} \|\sigma_{x_0} \nabla(u - \chi)\|_{L_\infty(\Omega)}.$$

In our a posteriori results, we wish to control  $\nabla(u - u_h)$  over any subset  $D$  of  $\Omega$ , so we define the piecewise constant weight

$$\sigma_D(T) = \frac{h_T}{\text{dist}(T, D) + h_T}.$$

We prove the localized a posteriori estimate

$$(1.5) \quad \frac{1}{C_1} \max_{T \in \mathcal{T}} \sigma_D(T) (\mathcal{E}_T - R_1(T)) \leq \|\sigma_D \nabla(u - u_h)\|_{L_\infty(\Omega)} \\ \leq C_2 \ell_{h,n} \max_{T \in \mathcal{T}} (\sigma_D(T) \mathcal{E}_T + \mathcal{E}_T^2) + R_3(\Omega).$$

Here  $C_2$  depends on the coefficients  $F_i$  and  $\|u\|_{W_\infty^2(\Omega)}$  in the nonlinear case, and the term  $\mathcal{E}_T^2$  may be dropped in the linear case. Note that the right-hand side of (1.5) also bounds  $\|\nabla(u - u_h)\|_{L_\infty(D)}$  since  $\sigma_D \equiv 1$  on  $D$ . Beyond the evaluation of standard residuals, the only requirement for the practical implementation of the estimator  $\max_{T \in \mathcal{T}} (\sigma_D(T) \mathcal{E}_T + \mathcal{E}_T^2)$  is the ability to efficiently compute the distance to the set  $D$ .

The constants  $C_1$  and  $C_2$  appearing in the estimates (1.4) and (1.5) depend on the unknown solution  $u$  in nonlinear situations, and these estimates are thus not strictly speaking a posteriori estimates. We do establish that  $C_1$  depends only on weak regularity properties of  $u$  ( $\|\nabla u\|_{L_\infty(\Omega)}$  and the Dini continuity of  $\nabla u$ ), and  $C_2$  depends on moderate regularity properties of  $u$  ( $\|u\|_{W_\infty^2(\Omega)}$ ). Although perhaps possible, tracing a more precise theoretical dependence of these constants on  $u$  would be difficult and, more important for practical purposes, unlikely to yield sharp results. In nonlinear problems especially, the estimators given here are therefore at most suitable for use as error indicators in adaptive mesh refinement.

We finally give a brief survey of relevant work related to pointwise a posteriori estimates and a posteriori estimates for nonlinear problems. In [Ver94], a general framework is given for a posteriori error estimation in canonical or energy norms for nonlinear problems. As mentioned in the introduction, our global estimates are maximum-norm analogues (in a rather more restricted situation) to the estimates presented in that work in that they yield reliable estimators only for  $u_h$  close enough to  $u$ , and the constants in the a posteriori estimates depend on the unknown solution  $u$ . The method of proof used here is partially inspired by that used in [Noc95] and [DDP00] to establish reliable and efficient a posteriori estimates for  $\|u - u_h\|_{L_\infty(\Omega)}$  for linear problems on general shape regular grids on arbitrary polygonal domains in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . A technique related to our localized estimates when  $D$  consists of a single point is the “dual weighted residual” method of [BR01], which involves solving a linear dual problem for each point for which one wishes to control the gradient error. Finally, in [HSWW01] and [SW04], localized a priori pointwise estimates are employed to provide sharply local and asymptotically exact pointwise control of the gradient via “gradient recovery” operators. These estimates have been shown to be valid only for smooth linear problems and on globally quasi-uniform meshes, however.

The outline of the paper is as follows. Section 2 contains further preliminaries and assumptions. In sections 3 and 4 we give precise results and proofs along with more detailed discussion of our global and localized a posteriori estimates, respectively. In section 5 we briefly discuss extensions to problems of the form (1.1).

**2. Preliminaries.** In this section we make a number of definitions and state some lemmas.

**2.1. Finite element approximation and mesh.** In addition to the notation and assumptions introduced in the previous section, we make the following definitions. By shape regular we mean that there exist positive constants  $r_1$  and  $r_2$  such that for each  $T \in \mathcal{T}$ , one may inscribe a sphere of radius  $r_1 h_T$  in  $T$  and inscribe  $T$  in a sphere of radius  $r_2 h_T$ . Letting  $T_x$  be an arbitrary element containing the point  $x$ , we denote by  $h(x)$  the quantity  $h_{T_x}$ . Additionally, we define the patches  $P_T =$

$\cup_{\{T' \in \mathcal{T} \text{ such that } \bar{T} \cap \bar{T}' \neq \emptyset\}} \bar{T}'$ ,  $P'_T = \cup_{\{T' \subset P_T\}} P_{T'}$ , and  $P''_T = \cup_{\{T' \subset P'_T\}} P_{T'}$ . Finally, we assume that there exists a finite element approximation  $u_h \in S_h$  to  $u$  satisfying

$$(2.1) \quad \int_{\Omega} \sum_{i=1}^n F_i(x, \nabla u_h) \chi_{x_i} \, dx = \int_{\Omega} f \chi \, dx \quad \forall \chi \in S_h.$$

The proof of our localized estimates requires a global growth condition on the mesh which is implied by shape regularity, a fact which we now formulate and prove.

**PROPOSITION 2.1.** *Assume the triangulation  $\mathcal{T}$  is shape regular. Then there exists a constant  $C_{\mathcal{T}}$  depending only on the shape regularity of  $\mathcal{T}$  such that for the barycenter  $x_T$  of each element  $T \in \mathcal{T}$ , there holds for each point  $y \in \Omega \setminus T$*

$$(2.2) \quad h(y) \leq C_{\mathcal{T}} |x_T - y|.$$

*Proof.* First fix an element  $T$  with barycenter  $x_T$ . Shape regularity implies that there exists  $0 < K_1$  such that

$$(2.3) \quad \text{dist}(x_T, \partial T) \geq K_1 h_T.$$

Next note that the elements contained in  $P_T$  are quasi-uniform, that is, there exist constants  $K_3 \leq 1 \leq K_4$  such that for each  $T' \subset P_T$ ,

$$K_3 h_T \leq h_{T'} \leq K_4 h_T.$$

We shall without loss of generality assume that  $K_4 \geq K_1$ . Finally, shape regularity implies that there exists  $0 < K_2 \leq 1$  such that for each point  $y \in T$ ,

$$(2.4) \quad B_{K_2 h_T}(y) \subset P_T.$$

We now assert that (2.2) holds with  $C_{\mathcal{T}} = \frac{K_4}{K_1 K_2}$ . Note from (2.3) that if  $y \in P_T \setminus T$ , then  $|y - x_T| \geq \text{dist}(x_T, \partial T) \geq K_1 h_T$ . Since  $K_2 \leq 1$ , we thus have

$$h(y) \leq K_4 h_T \leq \frac{K_4}{K_1} |y - x_T| \leq \frac{K_4}{K_1 K_2} |y - x_T|.$$

Now assume that  $y \notin P_T$ , that is,  $\bar{T} \cap \bar{T}_y = \emptyset$ . In order to reach a contradiction, we assume that  $h(y) > \frac{K_4}{K_1 K_2} |y - x_T|$ . Then since  $\frac{K_4}{K_1} \geq 1$ ,

$$K_2 h(y) > \frac{K_4}{K_1} |x_T - y| \geq |x_T - y|,$$

that is,  $x_T \in B_{K_2 h(y)}(y)$ . Thus by (2.4),  $x_T \in B_{K_2 h(y)}(y) \subset P_{T_y}$ , that is,  $\bar{T} \cap \bar{T}_y \neq \emptyset$ . This is a contradiction, so our proposition is proved.  $\square$

We shall also employ the Scott–Zhang interpolation operator  $I_h$  defined in [SZ90] which preserves homogeneous boundary conditions and satisfies

$$(2.5) \quad \|v - I_h v\|_{L_1(T)} \leq C h_T^{1+j} \|v\|_{W_1^{1+j}(P_T)}, \quad j = 1, 2,$$

and

$$(2.6) \quad \|v - I_h v\|_{W_1^1(T)} \leq C h_T^j \|v\|_{W_1^{1+j}(P_T)}, \quad j = 0, 1.$$

Here and throughout,  $C$  is a constant which depends at most on  $\Omega$  and the shape regularity of  $\mathcal{T}$ .

**2.2. Auxiliary problems and assumptions on coefficients.** We assume that the coefficients  $F_i(x, p)$  are twice continuously differentiable in  $p$  and define  $F_{ij}(x, p) = \frac{\partial}{\partial p_j} F_i(x, p)$  and  $F_{ijk}(x, p) = \frac{\partial^2}{\partial p_j \partial p_k} F_i(x, p)$ . We also require that  $F_i(x, p)$ ,  $1 \leq i \leq n$ , have derivatives with respect to the  $x$  variable which are uniformly bounded with respect to both  $x$  and  $p$ . We note that the analysis of our global results requires only that  $F_{ij}(x, p)$  be Dini-continuous with respect to the  $x$  variable, but  $F_i(x, p)$  must possess bounded spatial derivatives over each element in order to guarantee that the residual (1.3) is computable. Finally, we assume the ellipticity condition

$$(2.7) \quad \sum_{i,j=1}^n F_{ij}(x, p) \xi_i \xi_j > 0 \quad \forall \quad x \in \bar{\Omega}, \quad \xi \in \mathbb{R}^n \setminus \{0\}, p \in \mathbb{R}^n.$$

*Remark 2.2.* The conditions placed on the coefficients  $F_i$  may be slightly relaxed at the expense of some complication in our presentation. First, for our global results it is necessary not that  $F_{ijk}(x, p)$  exist but rather only that  $F_{ij}(x, p)$  be Hölder continuous in  $p$  with Hölder exponent  $0 < \alpha \leq 1$ . A perturbation term of the form  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)}^2$  arising in our global results would be replaced in this situation by  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)}^{1+\alpha}$ . Second, the ellipticity condition (2.7) must only hold for  $p \in \text{range}(\nabla u)$  and not for all  $p$  in  $\mathbb{R}^n$ . This latter observation would, for example, allow analysis of the  $\beta$ -Laplacian, where  $F_i(x, p) = p_i |p|^{\beta-2}$  and  $1 < \beta < \infty$ , if it could be established a priori that  $|\nabla u| \geq C > 0$ . For  $W_\infty^2$  solutions of the  $\beta$ -Laplacian with  $\beta > 2$ , Lemma 4.2 of [BL93] establishes such an inequality if  $|f| \geq C > 0$ . However, we are not aware of any regularity results in the literature which would guarantee such smooth solutions of this problem.

Two auxiliary linear problems are used in our analysis of quasilinear problems. Following for example [FR78], we define

$$a_{ij}^h = \int_0^1 F_{ij}(x, \nabla u_h + t \nabla(u - u_h)) \, dt, \quad i, j = 1, \dots, n,$$

and

$$a_{ij} = F_{ij}(x, \nabla u), \quad i, j = 1, \dots, n.$$

Correspondingly, we define bilinear forms

$$(2.8) \quad A_h(v, w) = \int_\Omega \sum_{i,j=1}^n a_{ij}^h v_{x_j} w_{x_i} \, dx$$

and

$$A(v, w) = \int_\Omega \sum_{i,j=1}^n a_{ij} v_{x_j} w_{x_i} \, dx.$$

From the ellipticity and smoothness of the coefficients  $F_i$  and the boundedness of  $\nabla u$ , we can conclude that  $[a_{ij}]$  is uniformly elliptic in  $\Omega$ , that is,

$$(2.9) \quad \lambda |\xi|^2 \leq \sum_{i,j=1}^n a_{ij} \xi_i \xi_j \leq \Lambda |\xi|^2.$$

We emphasize that for nonlinear problems,  $\lambda$  and  $\Lambda$  in general depend on  $\|\nabla u\|_{L^\infty(\Omega)}$ .

The ellipticity of  $A_h$ , on the other hand, depends upon  $\|\nabla u_h\|_{L^\infty(\Omega)}$ , but  $A_h$  satisfies the error equation

$$(2.10) \quad A_h(u - u_h, \chi) = \int_{\Omega} \sum_{i=1}^n (F_i(x, \nabla u) - F_i(x, \nabla u_h)) \chi_{x_i} \, dx = 0$$

for  $\chi \in S_h$ , and in fact for general  $v \in H_0^1(\Omega)$ ,

$$(2.11) \quad A_h(u - u_h, v) = \int_{\Omega} \sum_{i=1}^n (F_i(x, \nabla u) - F_i(x, \nabla u_h)) v_{x_i} \, dx.$$

We finally note that with  $S$  denoting the convex hull of  $\text{range}(\nabla u)$  and  $\text{range}(\nabla u_h)$ ,

$$(2.12) \quad \begin{aligned} |a_{ji} - a_{ji}^h| &= \left| \int_0^1 F_{ji}(\nabla u) - F_{ji}(\nabla u_h + t\nabla(u - u_h)) \, dt \right| \\ &\leq \int_0^1 \sum_{k=1}^n \|F_{jik}\|_{L^\infty(S)} (1-t) |\nabla(u - u_h)| \, dt \\ &\leq C_F |\nabla(u - u_h)|. \end{aligned}$$

The essential estimate  $\max_{1 \leq i, j, k \leq n} \|F_{ijk}\|_{L^\infty(S)} \leq C_F$  may be established here in one of two ways. It sometimes happens that  $F_{ijk}$  is bounded on  $\Omega \times \mathbb{R}^n$ , so that the bound is immediate and does not rely on  $\nabla u$  and  $\nabla u_h$ . If  $F_{ijk}$  is not globally bounded, then we must assume a priori that  $\|\nabla u_h\|_{L^\infty(\Omega)} \leq C$  or alternatively that  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)} \leq C$ .  $C_F$  can then be taken to be the bound for  $\max_{1 \leq i, j, k \leq n} |F_{ijk}(x, p)|$  on the compact set  $\{x \in \bar{\Omega}, |p| \leq \|\nabla u\|_{L^\infty(\Omega)} + C\}$ . Thus we shall assume that either  $F_{ijk}(x, p)$  is globally bounded in both  $x$  and  $p$  for all  $1 \leq i, j, k \leq n$  or that  $\|\nabla u_h\|_{L^\infty(\Omega)} \leq C$ .

**2.3. Green’s function estimates.** We denote by  $G(x, y)$  the Green’s function satisfying  $A(G(x, \cdot), v) = v(x)$  for sufficiently smooth  $v \in H_0^1(\Omega)$ . The following estimate for the first and mixed second derivatives of  $G$  is essential to our proofs.

LEMMA 2.3. *Assume that the coefficients  $a_{ij}$  are Dini-continuous and satisfy the uniform ellipticity condition (2.9), and let  $\Omega$  be smooth or convex. Assume that  $|\alpha| \leq 1$  and  $|\beta| \leq 1$ . Then for  $n \geq 3$*

$$(2.13) \quad |D_x^\alpha D_y^\beta G(x, y)| \leq C_G |x - y|^{2-n-|\alpha|-|\beta|}$$

and for  $n = 2$

$$(2.14) \quad |D_x^\alpha D_y^\beta G(x, y)| \leq C_G |x - y|^{2-n-|\alpha|-|\beta|} \log \frac{1}{|x - y|}.$$

Here  $C_G$  depends on  $\Omega$ , the Dini-continuity of the coefficients  $a_{ij}$ , and  $\lambda$  and  $\Lambda$ .

The estimate (2.13) for space dimension  $n \geq 3$  may be found in [GW82] assuming that  $\partial\Omega$  satisfies a uniform exterior sphere condition. This condition is met by both convex and smooth domains. The proof given in [GW82] does not carry directly over to  $n = 2$  due to the logarithmic nature of the singularity, but one may use the same method to obtain the suboptimal estimate (2.14) so long as the estimate

$$|G(x, y)| \leq C(\lambda, \Lambda, \Omega) \log \frac{1}{|x - y|}$$

is known. This estimate is contained in [DM95] under the weak restrictions of  $L^\infty$  and uniformly elliptic coefficients and Lipschitz boundary  $\partial\Omega$ . The suboptimal estimate (2.14) will only add an additional logarithmic factor to our results in the case  $n = 2$ .

**3. Global estimate.** In this section we state, discuss, and prove reliability and efficiency results for global estimators for  $\nabla(u - u_h)$ .

**3.1. Reliability of global estimators.** First we state the following upper bound for  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}$ .

**THEOREM 3.1.** *In addition to the assumptions of section 2, assume that  $u \in C^{1,\nu}(\bar{\Omega})$  for some  $0 < \nu \leq 1$ . Then for any  $0 < \alpha \leq \nu$  and any  $\beta \geq 1$ ,*

$$(3.1) \quad \|\nabla(u - u_h)\|_{L_\infty(\Omega)} \leq C_1 \beta^{\gamma(n)} \ell_{\underline{h},n} (\max_{T \in \mathcal{T}} \mathcal{E}_T + C_F \|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2) + C \underline{h}^{\alpha\beta} |u|_{C^{1,\alpha}(\bar{\Omega})}.$$

Here  $C_1$  depends on  $C_G$ , the ellipticity coefficients  $\lambda$  and  $\Lambda$ , and the shape regularity of  $\mathcal{T}$ . In the nonlinear case,  $C_1$  thus depends on  $\|\nabla u\|_{L_\infty(\Omega)}$ , the Dini-continuity of  $\nabla u$ , and the coefficients  $F_i$ . In the linear case,  $C_1$  does not depend on  $u$  and the term  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2$  in (3.1) does not appear.

*Remark 3.2.* The term  $\underline{h}^{\alpha\beta} |u|_{C^{1,\alpha}(\bar{\Omega})}$  may be omitted for  $\underline{h}$  small enough if we make the nondegeneracy assumption  $\underline{h}^\epsilon |u|_{C^{1,\alpha}(\bar{\Omega})} \leq \|\nabla(u - u_h)\|_{L_\infty(\Omega)}$  for some  $\epsilon > 0$ . We may then take  $\beta = (\epsilon + 1)/\alpha$  and  $\underline{h}$  small enough to kick back the resulting term  $C \underline{h} \|\nabla(u - u_h)\|_{L_\infty(\Omega)}$ . In [De04] we give a more precise nondegeneracy assumption which relies on lower bounds for polynomial approximations. In particular, assume that there exists a single point  $\tilde{x} \in \Omega$  and  $\eta > 0$  such that  $|D^2 u(\tilde{x})| \geq \tilde{C} > 0$  and  $\|u\|_{W_\infty^3(B_\eta(\tilde{x}))} \leq \tilde{C}'$ . The term  $\underline{h}^{\alpha\beta} |u|_{C^{1,\alpha}(\bar{\Omega})}$  may then be removed at the expense of a weak preasymptotic a priori dependence in the logarithmic factor. We do not give the details here. This more precise nondegeneracy assumption leads to an estimate which for linear problems is reliable on coarse meshes, and in most practical situations we may thus omit this term.

In [Noc95] and [DDP00], the Hölder continuity of  $u$  (instead of  $\nabla u$ ) and an assumption similar to the condition  $\underline{h}^\epsilon |u|_{C^{1,\alpha}(\bar{\Omega})} \leq \|\nabla(u - u_h)\|_{L_\infty(\Omega)}$  above were used in the establishment of asymptotically reliable residual estimators for  $\|u - u_h\|_{L_\infty(\Omega)}$  for linear problems on nonconvex polygonal domains. The technique we use in [De04] to remove the term  $\underline{h}^{\alpha\beta} |u|_{C^{1,\alpha}(\bar{\Omega})}$  is essentially a more rigorous and careful version of the argument contained in these works. A different and more sophisticated argument was used in [NSV03] to remove such nondegeneracy assumptions completely and thus prove  $L_\infty$  estimates which have no a priori dependence in the upper bounds and which are valid on coarse meshes. This technique does not appear to be applicable in the current context of  $W_\infty^1$  estimates, however.

*Remark 3.3.* The estimate (3.1) includes a logarithmic factor, whereas typical a priori estimates for  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}$  do not. It is possible to remove the logarithmic factor under the restriction that the mesh be quasi-uniform on balls of size  $c \log(1/\underline{h})$  for any fixed  $c > 0$ . Removing this computationally negligible factor would also lead to a stronger dependence of  $C_1$  on  $u$  in the nonlinear case or on the coefficients  $a_{ij}$  in the linear case.

Next note that we may kick back the term  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2$  in (3.1) if

$$(3.2) \quad C_1 C_F \beta^{\gamma(n)} \ell_{\underline{h},n} \|\nabla(u - u_h)\|_{L_\infty(\Omega)} \leq C^* < 1.$$

See [FR78] for an asymptotic a priori estimate for  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}$  on quasi-uniform meshes and under stricter regularity assumptions than we have made here. Using (3.2) and Remark 3.2 while noting that  $\|u_h\|_{L_\infty(\Omega)} \leq C$  if (3.2) holds, we may formulate an asymptotic reliability result which yields a computable estimator.

COROLLARY 3.4. Assume that  $\mathcal{T}$  is shape regular,  $u \in C^{1,\nu}(\bar{\Omega})$ ,  $\|\nabla(u-u_h)\|_{L_\infty(\Omega)}$  is small enough,  $\underline{h} \leq C$ , and that  $\|\nabla(u-u_h)\|_{L_\infty(\Omega)} \geq C\underline{h}^\epsilon |u|_{C^{1,\nu}(\bar{\Omega})}$  for some  $\epsilon > 0$ . Then

$$(3.3) \quad \|\nabla(u-u_h)\|_{L_\infty(\Omega)} \leq C_1 \left(\frac{\epsilon+1}{\nu}\right)^{\gamma(n)} \ell_{\underline{h},n} \max_{T \in \mathcal{T}} \mathcal{E}_T.$$

Here  $C_1$  is as in Theorem 3.1.

As stated in the introduction, the condition that  $\|\nabla(u-u_h)\|_{L_\infty(\Omega)}$  is small enough is an a priori and uncomputable condition. We refer again to [Ver94] for a posteriori estimates for nonlinear problems in integral norms which are reliable only for  $u_h$  close enough to  $u$  and which have reliability constants depending on the unknown solution  $u$ . For energy norms, an alternative approach to that of [Ver94] is to bound the error in a weighted problem-dependent norm depending on  $u_h$ . In some cases one may thus avoid the problem of unknown a priori constants replace the requirement that  $\nabla u$  and  $\nabla u_h$  be close enough with a computable condition whose fulfillment ensures reliability of the a posteriori upper bound; cf. [FV03] for such an example.

We finally note that our results may easily be combined with those of [Noc95] and [DDP00] to establish a bound for  $\|u-u_h\|_{L_\infty(\Omega)}$  for quasilinear problems on convex polyhedral domains in two and three space dimensions.

COROLLARY 3.5. Assume that the conditions of Theorem 3.1 are satisfied and that in addition the coefficients  $F_i$  are nonlinear and  $u \in W_\infty^2(\Omega)$ . Then for any  $\alpha > 0$  and  $\beta > 0$ ,

$$(3.4) \quad \begin{aligned} \|u-u_h\|_{L_\infty(\Omega)} &\leq \tilde{C} \beta^{\gamma(n)} \ell_{\underline{h}} [\max_{T \in \mathcal{T}} h_T \mathcal{E}_T + \max_{T \in \mathcal{T}} \mathcal{E}_T^2 \\ &\quad + \|\nabla(u-u_h)\|_{L_\infty(\Omega)}^4] + C[\underline{h}^{\alpha\beta} |u|_{C^\alpha(\bar{\Omega})} + \underline{h}^{2\alpha\beta} |u|_{C^{1,\alpha}(\bar{\Omega})}], \end{aligned}$$

where  $\tilde{C}$  depends on  $\|u\|_{W_\infty^2(\Omega)}$  and the coefficients  $F_i$ , and  $\ell_{\underline{h}}$  is a generic logarithmic factor.

Dropping the higher-order terms in the second line yields an asymptotically reliable estimator for  $\|u-u_h\|_{L_\infty(\Omega)}$ .

**3.2. Efficiency of global estimators.** Before stating our efficiency result, we define  $P_h$  to be the  $L_2$  projection onto the functions which are piecewise constant on  $\mathcal{T}$  and let  $\tilde{P}_h$  be the  $L_2$  projection onto the set of functions which are piecewise constant on the edges in  $\mathcal{T}$ .

THEOREM 3.6. Assume that either  $F_{ij}$  is globally bounded for  $1 \leq i, j \leq n$ , or that  $\|\nabla u_h\|_{L_\infty(\Omega)} \leq C$ . Then for any element  $T \in \mathcal{T}$ ,

$$(3.5) \quad \begin{aligned} \mathcal{E}_T &\leq \tilde{C}_1 \|\nabla(u-u_h)\|_{L_\infty(P_T)} + Ch_T \|\tilde{f}_h - P_h \tilde{f}_h\|_{L_\infty(P_T)} \\ &\quad + C \|[u_h] - \tilde{P}_h[u_h]\|_{L_\infty(\partial T)}. \end{aligned}$$

Here  $\tilde{f}_h(x) = f(x) + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(x, \nabla u_h)$  and  $\tilde{C}_1 = C \|a_{ij}^h\|_{L_\infty(\Omega)}$  is bounded independent of  $u_h$  under the assumptions of this theorem.

Remark 3.7. If the coefficients  $F_i(x, p)$  do not depend on  $x$  (as is the case for example for the prescribed mean curvature problem), then the higher-order term  $h_T \|\tilde{f}_h - P_h \tilde{f}_h\|_{L_\infty(P_T)} + \|[u_h] - \tilde{P}_h[u_h]\|_{L_\infty(\partial T)}$  reduces to  $\|f - P_h f\|_{L_\infty(P_T)}$ .

**3.3. Proof of reliability.** In our proofs we shall use a discrete  $\delta$ -function. Modifying the technique used in [Noc95], we fix a point  $x$  and define a function  $\delta_x$  as follows. Let  $x \in T \in \mathcal{T}$ . We then fix a simplex  $\tilde{T}$  such that  $x \in \tilde{T} \subset T$  and  $\tilde{T}$  is shape regular

with diameter  $\rho = h^\beta$  for  $\beta \geq 1$ , where  $\beta$  is as given in the statement of Theorem 3.1. We then let  $\delta_x \in C_0^\infty(\tilde{T})$  be a nonnegative function such that  $\int_{\tilde{T}} \delta_x \, dy = 1$ ,

$$(3.6) \quad \|\delta_x\|_{W_p^k(\tilde{T})} \leq C\rho^{n(1-1/p)-k},$$

and  $\text{dist}(\text{supp}(\delta_x), \partial\tilde{T}) \geq c\rho$  for some  $c > 0$ . Such a function  $\delta_x$  is easy to define by scaling and translation to  $\tilde{T}$  from a reference element.

Denote by  $\partial$  a first-order directional differential operator, that is,  $\partial = \nabla \cdot \vec{v}$  for some  $\vec{v}$  with  $|\vec{v}| = 1$ . Let  $x_0 \in T$  and  $\partial$  be such that  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)} \leq C|\partial(u - u_h)(x_0)|$ , and let  $\bar{\partial}u = \frac{1}{|\tilde{T}|} \int_{\tilde{T}} \partial u \, dx$ . Noting that  $\partial u_h$  is constant on  $\tilde{T}$  and that  $\bar{\partial}u = \partial u(x_1)$  for some  $x_1 \in \tilde{T}$ , we compute

$$(3.7) \quad \begin{aligned} \|\nabla(u - u_h)\|_{L_\infty(\Omega)} &\leq C|\partial(u - u_h)(x_0)| \\ &\leq C(|\partial u(x_0) - \partial u(x_1)| + |\bar{\partial}u - \partial u_h|) \\ &\leq C(\rho^\alpha |u|_{C^{1,\alpha}(\tilde{T})} + |(\bar{\partial}u - \partial u_h, \delta_{x_0})|) \\ &\leq C(\rho^\alpha |u|_{C^{1,\alpha}(\tilde{T})} + |(\bar{\partial}u - \partial u, \delta_{x_0})| + |(\partial(u - u_h), \delta_{x_0})|) \\ &\leq C(\rho^\alpha |u|_{C^{1,\alpha}(\tilde{T})} + \|\partial u(x_1) - \partial u\|_{L_\infty(\tilde{T})} + |(u - u_h, \partial\delta_{x_0})|) \\ &\leq C(\rho^\alpha |u|_{C^{1,\alpha}(\Omega)} + |(u - u_h, \partial\delta_{x_0})|) \\ &\leq C(h^{\alpha\beta} |u|_{C^{1,\alpha}(\Omega)} + |(u - u_h, \partial\delta_{x_0})|). \end{aligned}$$

We next introduce a discrete Green's function. With  $\partial$  and  $x_0$  as above, we define  $g^{x_0} \in H_0^1(\Omega)$  as the unique function satisfying

$$(3.8) \quad A(v, g^{x_0}) = (\partial\delta_{x_0}, v)$$

for all  $v \in H_0^1(\Omega)$ . Since the bound given below does not depend upon  $x_0$ , we shall suppress the dependence of  $g$  and  $\delta$  on  $x_0$  for the rest of this section. The heart of our proof consists of proving the following bound for  $\|g\|_{W_1^1(\Omega)}$ .

LEMMA 3.8. *If the conditions of Theorem 3.1 are satisfied, then*

$$(3.9) \quad \|g\|_{W_1^1(\Omega)} \leq C_g \beta^{\gamma(n)} \ell_{h,n},$$

where  $C_g = C(C_G, \lambda, \Lambda)$ .

In order to complete the proof of Theorem 3.1 given Lemma 3.8, we use (3.8), (2.10), and (2.11) to find that

$$(3.10) \quad \begin{aligned} (u - u_h, \partial\delta) &= A(u - u_h, g) = (A - A_h)(u - u_h, g) + A_h(u - u_h, g) \\ &= (A - A_h)(u - u_h, g) + A_h(u - u_h, g - I_h g) \\ &\leq C_F \|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2 \|g\|_{W_1^1(\Omega)} \\ &\quad + \left| \int_\Omega \sum_{i=1}^n (F_i(x, \nabla u) - F_i(x, \nabla u_h))(g - I_h g)_{x_i} \, dx \right|. \end{aligned}$$

Note that  $A_h = A$  if (1.2) is a linear problem, so the term  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2$  is dropped in this case as claimed. We use the easily-proven scaled trace inequality

$$\|v\|_{L_1(\partial T)} \leq C(h_T^{-1} \|v\|_{L_1(T)} + \|\nabla v\|_{L_1(T)})$$

and integrate the last term in (3.10) by parts elementwise to find

$$\begin{aligned}
& \left| \int_{\Omega} \sum_{i=1}^n (F_i(x, \nabla u) - F_i(x, \nabla u_h))(g - I_h g)_{x_i} \, dx \right| \\
&= \left| \sum_{T \in \mathcal{T}} \int_T \sum_{i=1}^n (F_i(x, \nabla u) - F_i(x, \nabla u_h))(g - I_h g)_{x_i} \, dx \right| \\
&= \left| \sum_{T \in \mathcal{T}} \int_T \left( - \sum_{i=1}^n \frac{\partial}{\partial x_i} (F_i(x, \nabla u) - F_i(x, \nabla u_h))(g - I_h g) \, dx \right. \right. \\
(3.11) \quad & \left. \left. + \int_{\partial T} \sum_{i=1}^n (F_i(x, \nabla u) - F_i(x, \nabla u_h)) n_i (g - I_h g) \, ds \right) \right| \\
&\leq \sum_{T \in \mathcal{T}} \|f + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(\cdot, \nabla u_h)\|_{L_{\infty}(T)} \|g - I_h g\|_{L_1(T)} \\
&\quad + \|[u_h]\|_{L_{\infty}(\partial T)} \|g - I_h g\|_{L_1(\partial T)} \\
&\leq \sum_{T \in \mathcal{T}} \left\| f + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(\cdot, \nabla u_h) \right\|_{L_{\infty}(T)} \|g - I_h g\|_{L_1(T)} \\
&\quad + \|[u_h]\|_{L_{\infty}(\partial T)} (h_T^{-1} \|g - I_h g\|_{L_1(T)} + \|\nabla(g - I_h g)\|_{L_1(T)}).
\end{aligned}$$

Finally, we apply the approximation results (2.5) and (2.6) and thus obtain

$$\begin{aligned}
& \sum_{T \in \mathcal{T}} \left\| f + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(\cdot, \nabla u_h) \right\|_{L_{\infty}(T)} \|g - I_h g\|_{L_1(T)} \\
&\quad + \|[u_h]\|_{L_{\infty}(\partial T)} (h_T^{-1} \|g - I_h g\|_{L_1(T)} + \|\nabla(g - I_h g)\|_{L_1(T)}) \\
(3.12) \quad &\leq \sum_{T \in \mathcal{T}} \mathcal{E}_T \|g\|_{W_1^1(P_T)} \\
&\leq C \|g\|_{W_1^1(\Omega)} \max_{T \in \mathcal{T}} \mathcal{E}_T.
\end{aligned}$$

Combining (3.12), (3.11), (3.10), and (3.7) and finally applying (3.9) completes the proof of Theorem 3.1 assuming Lemma 3.8.

To begin the proof of Lemma 3.8, we first note the elementary inequality

$$(3.13) \quad \|g\|_{H^1(\Omega)} \leq \frac{C(\Omega)}{\lambda} \|\delta\|_{L_2(\Omega)}.$$

In order to prove (3.9) for  $n \geq 3$ , we then note that if  $|x - x_0| > 2\rho$ , we may apply (2.13) and (3.6) to find that

$$\begin{aligned}
(3.14) \quad |\nabla_x g(x)| &= \left| \int_{\text{supp}(\delta)} \nabla_x G(x, y) \partial \delta(y) \, dy \right| \\
&= \left| \int_{\text{supp}(\delta)} \partial_y \nabla_x G(x, y) \delta \, dy \right| \\
&\leq C_G |x_0 - y|^{-n} \|\delta\|_{L_1(\Omega)} \leq C_G |x_0 - x|^{-n},
\end{aligned}$$

and similarly,

$$(3.15) \quad |g(x)| \leq C_G |x_0 - x|^{1-n}.$$

We then use (3.13), (3.6), (3.14), and (3.15) to compute

$$\begin{aligned} \|g\|_{W_1^1(\Omega)} &\leq C\rho^{n/2}\|g\|_{H^1(B_{3\rho}(x_0))} + \|g\|_{W_1^1(\Omega \setminus B_{3\rho}(x_0))} \\ &\leq \rho^{n/2} \frac{C(\Omega)}{\lambda} \|\delta\|_{L_2(\Omega)} + C_G \int_{C\rho}^{\text{diam}(\Omega)} r^{-n} r^{n-1} dr \\ &\leq \frac{C(\Omega)}{\lambda} \rho^{n/2} \rho^{-n/2} + C_G \log(1/\rho) \\ &\leq C \left( \frac{1}{\lambda} + C_G \right) \beta \log(1/h). \end{aligned}$$

If  $n = 2$ , we must apply (2.14) instead of (2.13). Since we always apply (2.14) with  $|x - y| \geq C\rho$ , this results in an extra factor of  $\log(1/\rho)$  in our estimates.  $\square$

*Proof of Corollary 3.5.* Letting  $\delta_{x_0} = \delta$  be as above, it is easy to compute that for some  $x_0 \in \Omega$ ,  $\|u - u_h\|_{L_\infty(\Omega)} \leq Ch^{\alpha\beta} |u|_{C^\alpha(\bar{\Omega})} + |(u - u_h, \delta)|$ . Let then  $\bar{g} \in H_0^1(\Omega)$  satisfy  $A(v, \bar{g}) = (v, \delta)$  for all  $v \in H_0^1(\Omega)$ . Then

$$(3.16) \quad \begin{aligned} |(u - u_h, \delta)| &= |A(u - u_h, \bar{g})| \leq |(A - A_h)(u - u_h, \bar{g})| + |A_h(u - u_h, \bar{g} - I_h \bar{g})| \\ &\leq C(\|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2 + \max_{T \in \mathcal{T}} h_T \mathcal{E}_T) \|\bar{g}\|_{W_1^2(\Omega)}. \end{aligned}$$

Inserting the bounds established in Theorem 3.1 of [Noc95] ( $n = 2$ ) and Corollary 2.3 of [DDP00] ( $n = 3$ ) for  $\|\bar{g}\|_{W_1^2(\Omega)}$  and also (3.1) into (3.16) yields (3.4).  $\square$

**3.4. Proof of efficiency.** We follow here the local argument given in [Ver89] and adapted to the maximum norm case in [Noc95] and [DDP00]. Recalling the definition of  $\tilde{f}_h$  from Theorem 3.6, we note first that for any  $v \in H_0^1(\Omega)$ ,

$$(3.17) \quad \sum_{T \in \mathcal{T}} \int_T \tilde{f}_h v \, dx + \frac{1}{2} \int_{\partial T} [u_h] v \, ds = A_h(u - u_h, v).$$

Now fix an element  $T$  and choose  $v = b_T$ , where  $b_T$  is the polynomial bubble function of degree  $n + 1$  which is obtained by multiplying the barycentric coordinates and scaling so that  $b_T$  is 1 at the barycenter of  $T$ . By transforming from a reference element, we see that  $\int_T b_T \, dx = Ch_T^n$ ,  $\|b_T\|_{L_1(T)} \leq Ch_T^n$ , and  $\|\nabla b_T\|_{L_1(T)} \leq Ch_T^{n-1}$ . Since  $b_T = 0$  on  $\partial T$ , we may thus compute from (3.17) that

$$\begin{aligned} \tilde{C}_1 h_T^{n-1} \|\nabla(u - u_h)\|_{L_\infty(T)} &\geq |A_h(u - u_h, b_T)| \\ &= \left| \int_T \tilde{f}_h b_T \, dx + \frac{1}{2} \int_{\partial T} [u_h] b_T \, ds \right| = \left| \int_T \tilde{f}_h b_T \, dx \right| \\ &= \left| \int_T (\tilde{f}_h - P_h \tilde{f}_h) b_T \, dx + P_h \tilde{f}_h|_T \int_T b_T \, dx \right| \\ &\geq Ch_T^n |P_h \tilde{f}_h|_T - C \|\tilde{f}_h - P_h \tilde{f}_h\|_{L_\infty(T)} \\ &\geq Ch_T^n (\|\tilde{f}_h\|_{L_\infty(T)} - \|\tilde{f}_h - P_h \tilde{f}_h\|_{L_\infty(T)}) - C \|\tilde{f}_h - P_h \tilde{f}_h\|_{L_\infty(T)} \\ &\geq Ch_T^n (\|\tilde{f}_h\|_{L_\infty(T)} - C \|\tilde{f}_h - P_h \tilde{f}_h\|_{L_\infty(T)}) \end{aligned}$$

so that

$$(3.18) \quad h_T \|\tilde{f}_h\|_{L_\infty(T)} \leq \tilde{C}_1 \|\nabla(u - u_h)\|_{L_\infty(T)} + Ch_T \|\tilde{f}_h - P_h \tilde{f}_h\|_{L_\infty(T)}.$$

Next let  $S = \bar{T} \cap \bar{T}'$  be a face of  $T$  not contained in  $\partial\Omega$ . We then define  $q_S$  to be the continuous piecewise polynomial of degree  $n$  which is 0 on  $\partial(T \cup T')$  and 1 at the barycenter of  $S$ . Note that  $\|q_S\|_{L_1(T \cup T')} \leq Ch_T^n$ ,  $\|\nabla q_S\|_{L_1(T \cup T')} \leq Ch_T^{n-1}$ ,  $\|q_S\|_{L_1(S)} \leq Ch_T^{n-1}$ , and  $\int_S q_S ds = Ch_T^{n-1}$ . Again computing using (3.17), we find that

$$\begin{aligned} \tilde{C}_1 h_T^{n-1} \|\nabla(u - u_h)\|_{L_\infty(T \cup T')} &\geq A_h(u - u_h, q_S) \\ &= \int_T \tilde{f}_h q_S \, dx + \int_S ([u_h] - \tilde{P}_h[u_h]) q_S ds + \int_S \tilde{P}_h[u_h] q_S ds \\ &\geq Ch_T^{n-1} \|[u_h]\|_{L_\infty(S)} \\ &\quad - C(h_T^n \|\tilde{f}_h\|_{L_\infty(T \cup T')} + h_T^{n-1} \|[u_h] - \tilde{P}_h[u_h]\|_{L_\infty(S)}) \end{aligned}$$

and

$$(3.19) \quad \begin{aligned} \|[u_h]\|_{L_\infty(S)} &\leq \tilde{C}_1 \|\nabla(u - u_h)\|_{L_\infty(P_T)} \\ &\quad + Ch_T \|\tilde{f}_h\|_{L_\infty(P_T)} + C \|[u_h] - \tilde{P}_h[u_h]\|_{L_\infty(S)}. \end{aligned}$$

Recalling that  $\mathcal{E}_T = h_T \|\tilde{f}_h\|_{L_\infty(T)} + \|[u_h]\|_{L_\infty(\partial T)}$  and combining (3.18) with (3.19) completes the proof of (3.5).  $\square$

**4. Localized estimates.**

**4.1. Reliability of localized estimators.** We first give an a posteriori bound for  $\|\sigma_D \nabla(u - u_h)\|_{L_\infty(\Omega)}$ .

**THEOREM 4.1.** *Let  $D \subset \Omega$ . In addition to the assumptions of section 2, assume that  $u \in C^{1,\nu}(\bar{\Omega})$  for some  $0 < \nu \leq 1$  if the coefficients  $F_i$  are linear and  $u \in W_\infty^2(\Omega)$  if the coefficients  $F_i$  are nonlinear. Then for any  $0 < \alpha \leq \nu$  and any  $\beta \geq 1$ ,*

$$(4.1) \quad \begin{aligned} \|\nabla(u - u_h)\|_{L_\infty(D)} &\leq \|\sigma_D \nabla(u - u_h)\|_{L_\infty(\Omega)} \\ &\leq \beta^{\gamma(n)} \ell_{h,n} [C_2 \max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T + C_1 C_F \|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2] \\ &\quad + Ch^{\alpha\beta} \max_{T \in \mathcal{T}} \sigma_D(T) |u|_{C^{1,\alpha}(\bar{T})}. \end{aligned}$$

Here  $C_1$  is as in Theorem 3.1 and  $C_2$  depends on  $C_G$ ,  $\|a_{ij}\|_{W_\infty^1(\Omega)}$ ,  $\lambda$ ,  $\Lambda$ , and the shape regularity of  $\mathcal{T}$ . In the nonlinear case,  $C_2$  thus depends on  $\|u\|_{W_\infty^2(\Omega)}$  and the coefficients  $F_i$ . In the linear case,  $C_2$  does not depend on  $u$  and the term  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2$  in (4.1) does not appear.

Note that the term  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2$  in (4.1) is not generally of higher order, in contrast to the situation which arises when the global estimate (3.1) is applied. One may insert (3.3) into (4.1) in the nonlinear case to yield the following asymptotic reliability result.

**COROLLARY 4.2.** *Assume that  $u \in W_\infty^2(\Omega)$ ,  $\|\nabla(u - u_h)\|_{L_\infty(\Omega)}$  and  $h$  are small enough and  $\|\sigma_D \nabla(u - u_h)\|_{L_\infty(\Omega)} \geq Ch^\epsilon \max_{T \in \mathcal{T}} \sigma_D(T) |u|_{C^{1,\nu}(\bar{T})}$  for some positive  $\nu$  and  $\epsilon$ . Then*

$$(4.2) \quad \begin{aligned} \|\nabla(u - u_h)\|_{L_\infty(D)} &\leq C_2 \left(\frac{1 + \epsilon}{\nu}\right)^{\gamma(n)} \ell_{h,n} \max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T + C_1^3 C_F \left(\frac{1 + \epsilon}{\nu}\right)^{2\gamma(n)} \ell_{h,n}^3 \max_{T \in \mathcal{T}} \mathcal{E}_T^2 \\ &\leq C(C_1, C_2, \epsilon) \ell_h \max_{T \in \mathcal{T}} (\sigma_D(T) \mathcal{E}_T + \mathcal{E}_T^2). \end{aligned}$$

Here  $C_1$  and  $C_2$  are as in Theorems 3.1 and 4.1,  $\ell_h$  is a generic logarithmic factor, and the term  $\mathcal{E}_T^2$  may be dropped in the linear case.

As in the case of our global estimator, the constants  $C_1$  and  $C_2$  potentially make it difficult to apply (4.2) efficiently and accurately as an error estimator in the nonlinear case. Even if we only wish to apply (4.2) as an error indicator, it likely will be necessary in most situations to gain some knowledge of the relative sizes of  $C_1$  and  $C_2$  as the terms in (4.2) could be weighted improperly otherwise. As stated in the introduction, a purely theoretical determination of these constants appears difficult, and their investigation is the subject of ongoing work.

One application of (4.2) is the computation of a gradient at a single point  $x_0 \in \Omega$  to within a given tolerance without requiring that  $\nabla u_h$  approximate  $\nabla u$  to the same tolerance globally (as would be the case if a global estimator were used). Here the localized estimate (4.2) is an alternative to the “dual weighted residual” approach of [BR01], which in this case essentially involves computing a finite element approximation to the discrete Green’s function  $g^{x_0}$  and inserting this approximation (using appropriate methods such as difference quotients to approximate second derivatives) into the appropriate residual equation, which for linear problems is

$$|\partial(u - u_h)(x_0)| \leq C \sum_{T \in \mathcal{T}} h_T \mathcal{E}_T |g^{x_0}|_{W_1^2(T)}.$$

Note that our localized analysis essentially involves bounding  $|g^{x_0}|_{W_1^2(T)}$  a priori instead of a posteriori as in the dual residual method. Since more of the work is done ahead of time, so to speak, localized estimators may be applied more easily and over larger subdomains than dual estimators, but potentially at the expense of some sharpness and unknown constants as compared with the dual weighted residual method. The advantages of localized estimators are their lower computational cost (the local nature of the discrete Green’s function is employed a priori instead of being computed a posteriori) and the fact that they can easily be applied over larger subdomains.

**4.2. Efficiency of localized estimators.** We shall show that our localized estimator is efficient (up to higher-order terms) in the linear case and in a certain sense also in the nonlinear case.

**THEOREM 4.3.** *Under the same conditions as are assumed in Theorem 3.6,*

$$(4.3) \quad \begin{aligned} \sigma_D(T) \mathcal{E}_T \leq & \tilde{C}_1 \|\sigma_D \nabla(u - u_h)\|_{L_\infty(P_T)} + C \|h \sigma_D(\tilde{f}_h - P_h \tilde{f}_h)\|_{L_\infty(P_T)} \\ & + C \|\sigma_D([u_h] - \tilde{P}[u_h])\|_{L_\infty(\partial T)}. \end{aligned}$$

Here  $\tilde{C}_1 = \|a_{ij}^h\|_{L_\infty(\Omega)}$ , and  $C$  only depends on  $\mathcal{T}$ .

*Proof.* To prove (4.3), we simply distribute the weight  $\sigma_D(T)$  through (3.5) while noting that  $h$  and  $\sigma_D$  are always equivalent on adjacent elements (and in particular on  $P_T$ ).  $\square$

*Remark 4.4.* In the linear case, (4.3) establishes immediately that up to higher-order terms,

$$\frac{1}{\tilde{C}_1} \max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T \leq \|\sigma_D \nabla(u - u_h)\|_{L_\infty(\Omega)} \leq C_2 \beta^{\gamma(n)} \ell_{h,n} \max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T.$$

In the nonlinear case, the perturbation term  $\max_{T \in \mathcal{T}} \mathcal{E}_T^2$  “morally” should behave as  $\|h_T \nabla(u - u_h)\|_{L_\infty(\Omega)}$ , which is bounded by  $\|\sigma_D \nabla(u - u_h)\|_{L_\infty(\Omega)}$ . However, one would have to resort to a priori estimates to prove such a statement. Instead, we combine the global reliability and efficiency estimates (3.1) and (3.5) with the localized estimates (4.2) and (4.3) while consolidating constants and ignoring higher-order terms to yield

the estimate

$$\begin{aligned} \frac{1}{\bar{C}} \max_{T \in \mathcal{T}} (\sigma_D(T) \mathcal{E}_T + \mathcal{E}_T^2) &\leq \|\sigma_D \nabla(u - u_h)\|_{L^\infty(\Omega)} + \|\nabla(u - u_h)\|_{L^\infty(\Omega)}^2 \\ &\leq \bar{C} \ell_h \max_{T \in \mathcal{T}} (\sigma_D(T) \mathcal{E}_T + \mathcal{E}_T^2). \end{aligned}$$

Thus the estimator  $\max_{T \in \mathcal{T}} (\sigma_D(T) \mathcal{E}_T + \mathcal{E}_T^2)$  reliably and efficiently estimates the quantity  $\|\sigma_D \nabla(u - u_h)\|_{L^\infty(\Omega)} + \|\nabla(u - u_h)\|_{L^\infty(\Omega)}^2$  instead of just the weighted norm  $\|\sigma_D \nabla(u - u_h)\|_{L^\infty(\Omega)}$  as originally intended.

**4.3. Proof of Theorem 4.1.** First we assume that  $D$  is a single point  $x_0 \in \Omega$ . We begin by picking a point  $x_1 \in \Omega$  and a first-order directional derivative  $\partial$  such that  $\|\sigma_{x_0} \nabla(u - u_h)\|_{L^\infty(\Omega)} \leq C \sigma_{x_0}(x_1) |\partial(u - u_h)(x_1)|$ . Here we have abused notation slightly by letting  $\sigma_{x_0}(x_1) = \sigma_{x_0}(T_{x_1})$ , where  $T_{x_1}$  is any element with  $x_1 \in \bar{T}_{x_1}$ . Proceeding as in (3.7) while noting that  $\|\sigma_{x_0}\|_{L^\infty(\Omega)} = 1$ , we obtain

$$(4.4) \quad \begin{aligned} \|\sigma_{x_0} \nabla(u - u_h)\|_{L^\infty(\Omega)} &\leq C \sigma_{x_0}(x_1) |\partial(u - u_h)(x_1)| \\ &\leq C \sigma_{x_0}(x_1) (|(u - u_h, \partial \delta_{x_1})| + \underline{h}^{\alpha\beta} |u|_{C^{1,\alpha}(\bar{T}_{x_1})}). \end{aligned}$$

We now compute as in (3.10) and (3.11) that

$$(4.5) \quad \begin{aligned} |(u - u_h, \partial \delta_{x_1})| &\leq |(A - A_h)(u - u_h, g^{x_1})| + |A_h(u - u_h, g^{x_1} - I_h g^{x_1})| \\ &\leq C_F \|\nabla(u - u_h)\|_{L^\infty(\Omega)}^2 \|g^{x_1}\|_{W_1^1(\Omega)} \\ &\quad + \sum_{T \in \mathcal{T}} \left[ \left\| f + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(x, \nabla u_h) \right\|_{L^\infty(T)} \|g^{x_1} - I_h g^{x_1}\|_{L_1(T)} \right. \\ &\quad \left. + \|[u_h]\|_{L^\infty(\partial T)} (h_T^{-1} \|g^{x_1} - I_h g^{x_1}\|_{L_1(T)} + \|\nabla(g^{x_1} - I_h g^{x_1})\|_{L_1(T)}) \right]. \end{aligned}$$

Next we note that by shape regularity, the elements in  $P_T''$  are quasi-uniform. Also, the weight  $\sigma_{x_1}$  is equivalent to 1 on  $P_{T_{x_1}}''$  and is always equivalent on adjacent elements. Using these facts, we then apply (2.2) along with (2.5) and (2.6) to obtain

$$(4.6) \quad \begin{aligned} &\sum_{T \in \mathcal{T}} \left\| f + \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(\cdot, \nabla u_h) \right\|_{L^\infty(T)} \|g^{x_1} - I_h g^{x_1}\|_{L_1(T)} \\ &\quad + \|[u_h]\|_{L^\infty(\partial T)} (h_T^{-1} \|g^{x_1} - I_h g^{x_1}\|_{L_1(T)} + \|\nabla(g^{x_1} - I_h g^{x_1})\|_{L_1(T)}) \\ &\leq \sum_{T \subset P_{T_{x_1}}''} \mathcal{E}_T \|g^{x_1}\|_{W_1^1(P_T)} \\ &\quad + \sum_{T \cap P_{T_{x_1}}'' = \emptyset} \frac{h_T}{\text{dist}(x_1, T) + h_T} \mathcal{E}_T (\text{dist}(x_1, T) + h_T) |g^{x_1}|_{W_1^2(P_T)} \\ &\leq C (\|g^{x_1}\|_{W_1^1(\Omega)} + \sum_{T \cap P_{T_{x_1}}'' = \emptyset} (\text{dist}(x_1, T) + h_T) |g^{x_1}|_{W_1^2(T)}) \\ &\quad \cdot \max_{T \in \mathcal{T}} \sigma_{x_1}(T) \mathcal{E}_T \\ &\leq C (\|g^{x_1}\|_{W_1^1(\Omega)} + \int_{\Omega \setminus P_{T_{x_1}}'} |x - x_1| |D^2 g^{x_1}| \, dx) \max_{T \in \mathcal{T}} \sigma_{x_1}(T) \mathcal{E}_T. \end{aligned}$$

We next state the fundamental lemma in the proof of our localized estimate.

LEMMA 4.5. *If the conditions of Theorem 4.1 are satisfied, then for any  $x_1 \in \Omega$ ,*

$$(4.7) \quad \int_{\Omega} |x - x_1| |D^2 g^{x_1}(y)| \, dx \leq C'_g \beta^{\gamma(n)} \ell_{h,n},$$

where  $C'_g$  depends on  $\lambda, \Lambda, \|a_{ij}\|_{W^1_{\infty}(\Omega)}$ , and the constant  $C_g$  from Lemma 3.8.

Assuming (4.7), we apply (3.9) and combine (4.4), (4.5), and (4.6) to find that

$$(4.8) \quad \begin{aligned} \|\sigma_{x_0} \nabla(u - u_h)\|_{L_{\infty}(\Omega)} &\leq \beta^{\gamma(n)} \ell_{h,n} [C'_g \max_{T \in \mathcal{T}} \sigma_{x_0}(T_{x_1}) \sigma_{x_1}(T) \mathcal{E}_T \\ &\quad + C_F C_g \|\nabla(u - u_h)\|_{L_{\infty}(\Omega)}^2] + C \max_{T \in \mathcal{T}} \sigma_D(T) |u|_{C^{1,\alpha}(\bar{T})}. \end{aligned}$$

In order to complete our proof, we thus must show that for  $T \in \mathcal{T}$ ,

$$(4.9) \quad \sigma_{x_0}(x_1) \sigma_{x_1}(T) \leq \sigma_{x_0}(T).$$

We shall compute with the weight  $\frac{h(x)}{|x_0 - x| + h(x)}$ , which is equivalent to but more convenient than  $\sigma_{x_0}(x)$ . Thus for any  $T \in \mathcal{T}$  and  $x_2 \in T$ ,

$$(4.10) \quad \begin{aligned} \sigma_{x_0}(x_1) \sigma_{x_1}(T) &\leq C \frac{h(x_1) h(x_2)}{(|x_0 - x_1| + h(x_1)) (|x_1 - x_2| + h(x_2))} \\ &= C \frac{h(x_2)}{|x_0 - x_2| + h(x_2)} \frac{h(x_1) (|x_0 - x_2| + h(x_2))}{(|x_0 - x_1| + h(x_1)) (|x_1 - x_2| + h(x_2))} \\ &\leq C \sigma_{x_0}(x_2) \frac{h(x_1) (|x_0 - x_2| + h(x_2))}{(|x_0 - x_1| + h(x_1)) (|x_1 - x_2| + h(x_2))}. \end{aligned}$$

Using the triangle inequality and noting from (2.2) that  $h(x_1) \leq C(|x_1 - x_2| + h(x_2))$ , we next compute that

$$(4.11) \quad \begin{aligned} &h(x_1) (|x_0 - x_2| + h(x_2)) \\ &\leq h(x_1) (|x_0 - x_1| + |x_1 - x_2| + h(x_2)) \\ &= h(x_1) |x_0 - x_1| + h(x_1) (|x_1 - x_2| + h(x_2)) \\ &\leq C (|x_1 - x_2| + h(x_2)) |x_0 - x_1| + h(x_1) (|x_1 - x_2| + h(x_2)). \end{aligned}$$

Noting that the expression above is bounded by  $C$  times the denominator of (4.10), we combine (4.10) and (4.11) to obtain (4.9). Inserting (4.9) into (4.8) completes the proof of (4.1) for  $D = x_0$  assuming Lemma 4.5. Taking the maximum of (4.1) over  $x_0 \in D$  while recalling (4.9) completes the proof of (4.1) for arbitrary  $D \subset \Omega$ .

In order to prove Lemma 4.5, we shall need the linear  $H^2_2$  regularity result

$$(4.12) \quad \|g\|_{H^2_2(\Omega)} \leq C_{reg} \|\partial \delta\|_{L_2(\Omega)},$$

where  $C_{reg} = C(\lambda, \Lambda, \|a_{ij}\|_{W^1_{\infty}(\Omega)})$ . This result is standard for smooth domains and may be found in [Gr85] for convex (including convex polyhedral) domains. Here and in what follows we suppress the dependence of  $g$  and  $\delta$  on  $x_1$ .

We now decompose  $\Omega$  into dyadic annuli. Let  $\Omega_0 = B_{3\rho}(x_1)$ , so that according to our definitions  $\text{dist}(\text{supp}(\delta), \partial \Omega_0) > C\rho$ . We then define  $d_j = 2^j 3\rho$ ,  $j = 0, \dots, N$ ,  $\tilde{\Omega}_j = \{x \in \mathbb{R}^n \text{ such that } d_{j-1} \leq |x - x_1| \leq d_j\}$ ,  $\Omega_j = \tilde{\Omega}_j \cap \Omega$ , and  $\Omega'_j = \Omega_{j-1} \cup \Omega_j \cup \Omega_{j+1}$ . Note that  $\Omega = \bigcup_{j=0}^N \Omega_j$  with  $N \leq C \log(1/\rho)$ . Finally, we let  $\omega_j \in C^{\infty}_0(\tilde{\Omega}_{j-1} \cup \tilde{\Omega}_j \cup \tilde{\Omega}_{j+1})$  be a cutoff function which is 1 on  $\tilde{\Omega}_j$  and which satisfies

$\|\omega_j\|_{W_\infty^k(\Omega)} \leq C d_j^{-k}$ ,  $k = 0, 1, 2$ . Then

$$\begin{aligned}
(4.13) \quad \int_{\Omega} |x - x_1| |D^2 g^{x_1}| \, dx &\leq C d_1^{n/2+1} \|D^2 g\|_{L_2(\Omega'_1)} + \sum_{i=2}^N d_i \|D^2 g\|_{L_1(\Omega_i)} \\
&\leq C_{reg} \rho^{n/2+1} \|\partial \delta\|_{L_2(\Omega)} + \sum_{i=2}^N d_i \|D^2(\omega_j g)\|_{L_1(\Omega)} \\
&\leq C_{reg} + \sum_{i=2}^N d_i \|D^2(\omega_j g)\|_{L_1(\Omega)}.
\end{aligned}$$

Abusing notation slightly by letting  $A$  denote the matrix of coefficients  $[a_{ij}]$ , we compute that for  $v \in H_0^1(\Omega)$  and  $j > 1$ ,

$$\begin{aligned}
A(v, \omega_j g) &= (v, -\operatorname{div}(A^* \nabla(\omega_j g))) \\
&= (v, -\operatorname{div}(A^*(g \nabla \omega_j + \omega_j \nabla g))) \\
&= (v, -g \operatorname{div}(A^* \nabla \omega_j) - A^* \nabla \omega_j \cdot \nabla g - A^* \nabla g \cdot \nabla \omega_j - \omega_j \operatorname{div}(A^* \nabla g)) \\
&= (v, -g \operatorname{div}(A^* \nabla \omega_j) - A^* \nabla \omega_j \cdot \nabla g - A^* \nabla g \cdot \nabla \omega_j - \omega_j \delta) \\
&= (v, -g \operatorname{div}(A^* \nabla \omega_j) - A^* \nabla \omega_j \cdot \nabla g - A^* \nabla g \cdot \nabla \omega_j)
\end{aligned}$$

since  $\omega_j$  and  $\delta$  have disjoint support for  $j > 1$ . Then applying the regularity result (4.12) to  $\omega_j g$ , we find that

$$(4.14) \quad \|D^2(\omega_j g)\|_{L_2(\Omega)} \leq C_{reg} \|a_{ij}\|_{W_\infty^1(\Omega)} \left( \frac{1}{d_j^2} \|g\|_{L_2(\Omega'_j)} + \frac{1}{d_j} \|\nabla g\|_{L_2(\Omega'_j)} \right).$$

We then insert (4.14) into (4.13) while recalling (3.14) and (3.15) to find that for  $n \geq 3$ ,

$$\begin{aligned}
(4.15) \quad \int_{\Omega} |x - x_1| |D^2 g^{x_1}| \, dx &\leq C(C_{reg} + \sum_{i=1}^N d_i^{n/2+1} \left( \frac{1}{d_j^2} \|g\|_{L_2(\Omega_j)} + \frac{1}{d_j} \|\nabla g\|_{L_2(\Omega_j)} \right)) \\
&\leq C_{reg} \|a_{ij}\|_{W_\infty^1(\Omega)} \left( 1 + \sum_{i=1}^N (d_j^{n-1} \|g\|_{L_\infty(\Omega_j)} + d_j^n \|\nabla g\|_{L_\infty(\Omega_j)}) \right) \\
&\leq C_{reg} \|a_{ij}\|_{W_\infty^1(\Omega)} C_G \left( 1 + \sum_{i=1}^N (d_j^{n-1} d_j^{1-n} + d_j^n d_j^{-n}) \right) \\
&\leq C_{reg} \|a_{ij}\|_{W_\infty^1(\Omega)} C_G (1 + \log(1/\rho)) \\
&\leq C_{reg} \|a_{ij}\|_{W_\infty^1(\Omega)} C_G \beta \log(1/h).
\end{aligned}$$

When  $n = 2$ , an extra factor of  $\log(1/\rho)$  enters the estimate (4.15) as before. Thus the proof of Lemma 4.5 is completed.  $\square$

**5. Extension of results to the general quasilinear equation (1.1).** In this section we outline the steps necessary to extend our results to operators of the form (1.1). We first consider the treatment of nonhomogeneous Dirichlet conditions in a model problem of the form (1.2), then we consider general operators of the form (1.1) with homogeneous boundary conditions.

**5.1. Nonhomogeneous boundary conditions.** We consider here the model problem (1.2), but now with the more general Dirichlet boundary condition  $u = b$  on  $\partial\Omega$  for some  $b \in W_\infty^1(\Omega)$ . We also assume that  $b_h$  is a piecewise linear finite element approximation to  $b$  and that  $u_h$  with  $u_h - b_h \in S_h$  solves (2.1). The following is a corollary to Theorem 3.1 and Theorem 4.1.

COROLLARY 5.1. *Under the conditions of Theorem 3.1,*

$$(5.1) \quad \|\nabla(u - u_h)\|_{L_\infty(\Omega)} \leq C_1 \beta^{\gamma(n)} \ell_{h,n} [\max_{T \in \mathcal{T}} \mathcal{E}_T + C_F \|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2 + \|\nabla(b - b_h)\|_{L_\infty(\Omega)}] + C \underline{h}^{\alpha\beta} |u|_{C^{1,\alpha}(\bar{\Omega})}.$$

*Under the conditions of Theorem 4.1,*

$$(5.2) \quad \begin{aligned} & \|\nabla(u - u_h)\|_{L_\infty(D)} \\ & \leq \beta^{\gamma(n)} \ell_{h,n} [C_2 \max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T + C_1 C_F \|\nabla(u - u_h)\|_{L_\infty(\Omega)}^2] \\ & \quad + C_1 \|(b - b_h)[\text{dist}(\cdot, D) + \underline{h}^\beta]^{-n}\|_{L_1(\partial\Omega)} \\ & \quad + C \underline{h}^{\alpha\beta} \max_{T \in \mathcal{T}} \sigma_D(T) |u|_{C^{1,\alpha}(\bar{T})}. \end{aligned}$$

*Sketch of Proof.* We proceed as in (3.7) through (3.10), then let  $\vec{n}$  be the outward normal on  $\partial\Omega$  and compute that for  $\delta$  and  $g$  defined with respect to the point  $x_0$ ,

$$(5.3) \quad (u - u_h, \partial\delta) = A(u - u_h, g) - \int_{\partial\Omega} (b - b_h)(A\nabla g \cdot \vec{n}) \, d\sigma.$$

To prove (5.1), we bound  $A(u - u_h, g)$  precisely as before and compute

$$(5.4) \quad \left| \int_{\partial\Omega} (b - b_h)(A\nabla g \cdot \vec{n}) \, d\sigma \right| = |(b - b_h, \partial\delta) - A(b - b_h, g)| \\ \leq |(\partial(b - b_h), \delta)| + \Lambda \|\nabla(b - b_h)\|_{L_\infty(\Omega)} \|\nabla g\|_{L_1(\Omega)}.$$

and then apply (3.6) with  $p = 1$  and  $k = 1$  along with Lemma 3.8.

In order to prove (5.2), we let  $x_0 \in D$  be such that  $\|\nabla(u - u_h)\|_{L_\infty(D)} = |\nabla(u - u_h)(x_0)|$ . Recall that  $\delta_{x_0}$  may always be defined so that  $\text{dist}(\text{supp}(\delta), \partial\Omega) \geq c\rho$ . A calculation similar to (3.14) then yields  $|(A\nabla g \cdot \vec{n})(y)| \leq C_1 [|\rho + |x_0 - y||]^{-n}$  for  $y \in \partial\Omega$ . Inserting this inequality into (5.3), recalling that  $\rho = \underline{h}^\beta$ , and bounding  $A(u - u_h, g)$  as in (4.5) and following completes the proof.  $\square$

In [DR98] an a posteriori energy-norm bound is given which treats Dirichlet data in a fashion similar to (5.1). The term  $\|(b - b_h)[\text{dist}(\cdot, D) + \underline{h}^\beta]^{-n}\|_{L_1(\partial\Omega)} \leq Ch^{-1} \|b - b_h\|_{L_\infty(\partial\Omega)}$  in (5.2) is very similar to one appearing in the a priori estimates given in Theorem A.1 of [BTW03]. One may easily compute that

$$(5.5) \quad \|(b - b_h)[\text{dist}(\cdot, D) + \underline{h}^\beta]^{-n}\|_{L_1(\partial\Omega)} \leq C \min(\text{dist}(D, \partial\Omega)^{-1}, \underline{h}^{-\beta}) \|b - b_h\|_{L_\infty(\partial\Omega)}.$$

If  $D \subset\subset \Omega$  this term is thus of higher order, reflecting the localization of the error to  $D$ . If  $D$  abuts  $\partial\Omega$ , however, the term  $\underline{h}^\beta$  leads to suboptimality if  $\beta > 1$ . (Note that this problem is not encountered in the a priori estimates of [BTW03] on quasi-uniform meshes, where  $\underline{h}^\beta$  may be replaced by the mesh size  $h$ .) One may in this case instead estimate the error in approximating the Dirichlet data by  $\|\nabla(b - b_h)\|_{L_\infty(\Omega)}$  as in (5.1), but this estimate does not reflect the more local nature of the error. Thus (5.2) could likely be improved, although it appears difficult to do so using the present techniques.

**5.2. Theoretical comments on more general operators.** We assume that  $u$  solves (1.1) with  $u = 0$  on  $\partial\Omega$ . As before, we assume  $\Omega$  to be convex and polygonal. We also assume that  $u_h \in S_h$  satisfies

$$\int_{\Omega} \sum_{i=1}^n F_i(x, u_h, \nabla u_h) v_{h,x_i} + F_0(x, u_h, \nabla u_h) v_h \, dx = 0, \quad v_h \in S_h.$$

The essential linear auxiliary operators  $A$  and  $A_h$  defined in section 2.2 may be easily modified to aid in the analysis of problems of the form (1.1). Letting  $F_{j0} = \frac{\partial}{\partial z} F_j(x, z, p)$  and  $v_{x_0} = v$ , we have for  $i = 0, \dots, n$  that

$$\begin{aligned} &F_i(x, u, \nabla u) - F_i(x, u_h, \nabla u_h) \\ &= \sum_{j=0}^n \int_0^1 F_{ij}(x, u_h + t(u - u_h), \nabla u_h + t\nabla(u - u_h))(u - u_h)_{x_j} \, dt. \end{aligned}$$

For  $0 \leq i, j \leq n$ , we then make the definitions

$$\begin{aligned} a_{ij}^h(x) &= \int_0^1 F_{ij}(x, u_h + t(u - u_h), \nabla(u_h + t\nabla(u - u_h))) \, dt, \\ a_{ij}(x) &= F_{ij}(x, u, \nabla u), \\ A_h(v, w) &= \int_{\Omega} \sum_{i,j=0}^n a_{ij}^h(x) v_{x_j} w_{x_i} \, dx, \\ A(v, w) &= \int_{\Omega} \sum_{i,j=0}^n a_{ij}(x) v_{x_j} w_{x_i} \, dx. \end{aligned}$$

Note also that

$$(5.6) \quad |a_{ij}(x) - a_{ij}^h(x)| \leq \sum_{k=0}^n \|F_{ijk}\|_{L^\infty} |(u - u_h)_{x_k}(x)|.$$

$A$  and  $A_h$  as defined here differ from their previous incarnations mainly in that some lower-order terms are now included. (Note that summation indices now run from 0 to  $n$  instead of from 1 to  $n$ .) Finally, the residual  $\mathcal{E}_T$  must be modified to reflect the presence of lower-order terms. Thus we now define

$$\mathcal{E}_T = h_T \left\| \sum_{i=1}^n \frac{\partial}{\partial x_i} F_i(\cdot, u_h, \nabla u_h) - F_0(\cdot, u_h, \nabla u_h) \right\|_{L^\infty(T)} + \|[u_h]\|_{L^\infty(\partial T)}.$$

The analytical assumptions of section 2 must be modified only slightly. We must still assume that the operator  $A$  is uniformly elliptic in  $\Omega$ , a fact which may be established exactly as in section 2.2. Second, we must assume that  $A$  admits unique and sufficiently regular solutions for homogeneous Dirichlet problems. Note that establishing existence and uniqueness of solutions of such problems is potentially complicated by the presence of lower-order terms. In nonlinear problems, we must as before assume some regularity of  $u$  as well, although lower regularity may be required of  $u$  in the case of mildly nonlinear problems, as we show below. Next, the Green's function estimates of Lemma 2.3 must hold. [GW82] states such results only for divergence form operators with no lower-order terms, although the same techniques should apply when lower-order terms are present. Finally, the constant  $C_F$  arising in (2.12) must be bounded as before.

**5.3. Example: A mildly nonlinear problem.** Consider the mildly nonlinear problem

$$(5.7) \quad \begin{aligned} - \sum_{i,j=0}^n \frac{\partial}{\partial x_i} (\tilde{a}_{ij}(x, u) \frac{\partial u}{\partial x_j}) &= f(x) \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

where we assume that the coefficients  $F_i(x, u, \nabla u) = \sum_{j=1}^n \tilde{a}_{ij}(x, u) u_{x_j}$  satisfy the requirements outlined in the previous subsection. Note that the term  $(A - A_h)(u - u_h, g)$  appearing in (3.10) and (4.5) of the proofs of Theorem 3.1 and Theorem 4.1 led there to a nonlinear perturbation error term of the form  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)}^2$ . Since  $F_i(x, u, \nabla u)$  now depends only linearly on  $\nabla u$ , however, (5.6) yields

$$|a_{ij} - a_{ij}^h| \leq C_F |u - u_h|$$

so that

$$|(A - A_h)(u - u_h, g)| \leq C \|u - u_h\|_{L^\infty(\Omega)} \|\nabla(u - u_h)\|_{L^\infty(\Omega)} \|g\|_{W_1^1(\Omega)}.$$

Thus the nonlinear perturbation term is now of higher order than it generally is for approximations of  $u$  solving (1.1).

Using this observation, we may obtain estimates similar to, but often simpler than, those in Theorem 3.1, Corollary 3.5, and Theorem 4.1. We assume here that a nondegeneracy condition as outlined in Remark 3.2 is satisfied and that  $u$  possess sufficient regularity. First, analogous to Theorem 3.1 and Corollary 3.4, we find that if  $\|u - u_h\|_{L^\infty(\Omega)}$  is small enough, then

$$\begin{aligned} \|\nabla(u - u_h)\|_{L^\infty(\Omega)} &\leq C_3 \ell_{h,n} (\max_{T \in \mathcal{T}} \mathcal{E}_T + C_F \|u - u_h\|_{L^\infty(\Omega)} \|\nabla(u - u_h)\|_{L^\infty(\Omega)}) \\ &\leq 2C_3 \ell_{h,n} \max_{T \in \mathcal{T}} \mathcal{E}_T. \end{aligned}$$

Analogous to Corollary 3.5, we obtain for  $\|\nabla(u - u_h)\|_{L^\infty(\Omega)}$  small enough that

$$(5.8) \quad \begin{aligned} \|u - u_h\|_{L^\infty(\Omega)} &\leq C_4 \ell_h (\max_{T \in \mathcal{T}} h_T \mathcal{E}_T + C_F \|u - u_h\|_{L^\infty(\Omega)} \|\nabla(u - u_h)\|_{L^\infty(\Omega)}) \\ &\leq 2C_4 \ell_h \max_{T \in \mathcal{T}} h_T \mathcal{E}_T. \end{aligned}$$

Finally, we employ (5.8) and note that  $h_T \leq C \sigma_D(T)$  for  $D \subset \Omega$  to find that for  $\|\nabla(u - u_h)\|$  small enough,

$$(5.9) \quad \begin{aligned} \|\sigma_D \nabla(u - u_h)\|_{L^\infty(\Omega)} &\leq C_5 \ell_{h,n} (\max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T + C_F \|\nabla(u - u_h)\|_{L^\infty(\Omega)} \|u - u_h\|_{L^\infty(\Omega)}) \\ &\leq C_5 \ell_{h,n} (\max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T + C_F \|\nabla(u - u_h)\|_{L^\infty(\Omega)} C_4 \ell_h \max_{T \in \mathcal{T}} h_T \mathcal{E}_T) \\ &\leq C_5 \ell_{h,n} \max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T (1 + C_F \|\nabla(u - u_h)\|_{L^\infty(\Omega)} C_4 \ell_h) \\ &\leq 2C_5 \ell_{h,n} \max_{T \in \mathcal{T}} \sigma_D(T) \mathcal{E}_T. \end{aligned}$$

Since the coefficients of the dual linear operator  $A$  now have essentially the same regularity as  $u$  instead of as  $\nabla u$ , the constants  $C_3$ ,  $C_4$ , and  $C_5$  above depend more weakly on the regularity of  $u$  than do the corresponding constants in Theorem 3.1, Corollary 3.5, and Theorem 4.1. Indeed,  $C_3$  depends on  $\|u\|_{L^\infty(\Omega)}$  and the Dini-continuity of  $u$  as opposed to  $C_1$  from Theorem 3.1, which depends on  $\|\nabla u\|_{L^\infty(\Omega)}$

and the Dini-continuity of  $\nabla u$ .  $C_4$  and  $C_5$  depend only on  $\|u\|_{W_\infty^1(\Omega)}$  as opposed to  $\tilde{C}$  and  $C_2$  from Corollary 3.5 and Theorem 4.1, which both depend on  $\|u\|_{W_\infty^2(\Omega)}$ . We also note that (5.8) and (5.9) only require that  $u \in C^{1,\nu}(\bar{\Omega})$  for some  $\nu > 0$  in order to hold, whereas the corresponding estimates in Corollary 3.5 and Theorem 4.1 require  $u \in W_\infty^2(\Omega)$ .

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