

CONVERGENCE OF AN ADAPTIVE FINITE ELEMENT METHOD FOR CONTROLLING LOCAL ENERGY ERRORS *

ALAN DEMLOW[†]

Abstract. A number of works concerning rigorous convergence theory for adaptive finite element methods (AFEM) for controlling global energy errors have appeared in recent years. However, many practical situations demand AFEM designed to efficiently compute quantities which depend on the unknown solution only on some subset of the overall computational domain. In this work we prove convergence of an adaptive finite element method for controlling local energy errors. The first step in our convergence proof is the construction of novel a posteriori error estimates for controlling a weighted local energy error. This weighted local energy notion admits versions of standard ingredients for proving convergence of AFEM such as quasi-orthogonality and error contraction, but modulo “pollution terms” which use weaker norms to measure effects of global solution properties on the local energy error. We then prove several convergence results for AFEM based on various marking strategies, including a contraction result in the case of convex polyhedral domains.

Key words. Adaptive finite element methods; convergence of adaptive finite element methods; local error estimates.

AMS subject classification. 65N15, 65N30

1. Introduction and Results. Consider the elliptic model problem

$$\begin{aligned} -\Delta u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$, $n = 2, 3$, is a polyhedral domain, and where for technical reasons we assume that $f \in L_p(\Omega)$ for some $p > 4$. The results presented here can be generalized to other linear elliptic operators, but we present the simplest case in order to focus on essential ideas.

In this work we present the first convergence proof for an adaptive finite element method (AFEM) for controlling local energy errors. An AFEM is an iterative feedback procedure of the form

$$\text{solve} \rightarrow \text{estimate} \rightarrow \text{mark} \rightarrow \text{refine}. \tag{1.2}$$

Such adaptive algorithms have for many years been a standard tool for efficiently approximating solutions to partial differential equations such as (1.1). The convergence properties of AFEM have become the subject of intense theoretical study only in the past few years, however. We refer to [12], [20], [19], [7], [22], and [21] for an overview of progress in basic convergence theory for AFEM for linear elliptic problems.

A common feature of almost all results published to date concerning convergence of AFEM is that the error estimated in the “estimate” step in (1.2) is the global energy error. In contrast, the output “quantity of interest” in many practical situations is related to some other norm or functional of the solution which may depend on u only in some subset D of Ω . In such cases it is desirable to construct an AFEM that takes advantage of the local nature of the desired output by directing refinement toward the region of interest. Such algorithms must also control “pollution effects” from outside

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[†]Department of Mathematics, University of Kentucky, 715 Patterson Office Tower, Lexington, KY 40506-0027 (demlow@ms.uky.edu).

of D . That is, some global refinement is generally necessary even when only local information is desired.

Here we study an adaptive algorithm based on explicit a posteriori estimates for controlling global pollution effects on the local energy error. Similar estimators have been previously considered in the literature (cf. [27], [17], [9]), but we make some essential modifications that are necessary for proving convergence of the resulting AFEM. Other AFEM for controlling pollution effects have also been proposed, especially “goal-oriented” adaptive algorithms in which a computed dual solution is used to encode the desired error notion (cf. [5], [15], [14]). In contrast to these duality-based estimators, explicit estimators readily yield rigorous a posteriori upper bounds on the local energy error. Explicit estimators also form the basis for certain approaches to parallel adaptive computations (cf. [27], [3]), and in fact the construction of provably convergent versions of such algorithms is one potential application of our results. The latter application also leads us to consider in detail the effects of AFEM which mark using *only* local energy error indicators.

Let $\{\mathcal{T}_i\}$ ($i \geq 0$) be a nested sequence of regular triangulations of Ω . Let $S_i \subset H_0^1(\Omega)$ be a standard Lagrange finite element space on \mathcal{T}_i , and let $u_i \in S_i$ satisfy

$$A(u_i, v_i) := \int_{\Omega} \nabla u_i \cdot \nabla v_i \, dx = \int_{\Omega} f v_i \, dx, \quad v_i \in S_i. \quad (1.3)$$

For any $U \subset \Omega$, let $\|v\|_U := \sqrt{\int_U |\nabla v|^2 \, dx}$ denote the local energy (semi-)norm.

Nitsche and Schatz proved local a priori energy error estimates in the classical work [23] (cf. [10] for a proof in the case of adapted grids). Let D be a subdomain of Ω , and for a fixed $d > 0$ let $D_d = \{x \in \Omega : \text{dist}(x, D) < d\}$. Then

$$\|u - u_i\|_D \leq C \min_{\chi \in S_i} \|u - \chi\|_{D_d} + \frac{C}{d} \|u - u_i\|_{L_2(D_d)}. \quad (1.4)$$

The local energy error is thus split into a local almost-best-approximation term of optimal order plus a global “pollution term” which measures the influence of global solution properties upon the local solution quality in a weaker norm. (1.4) may be roughly restated as follows.

A priori heuristic for local energy errors: *Local finite element energy errors behave like global finite element energy errors, up to pollution terms.*

[27] and [17] contain local a posteriori estimates that are similar in spirit. For $T \in \mathcal{T}_i$, let $\eta_{1,i}(T)^2 = h_T^2 \|f + \Delta u_i\|_{L_2(T)}^2 + h_T \|\llbracket \nabla u_i \rrbracket\|_{L_2(\partial T)}^2$ be the standard energy residual indicator; here $\llbracket \nabla u_i \rrbracket$ is the jump in normal derivative across ∂T and $h_T = \text{diam}(T)$. Then

$$\|u - u_i\|_D \leq C \left(\sum_{T \cap D_d \neq \emptyset} \eta_{1,i}(T)^2 \right)^{1/2} + \frac{C}{d} \|u - u_i\|_{L_2(D_d)}. \quad (1.5)$$

Finally obtaining an a posteriori estimate from (1.5) requires bounding the pollution error $\|u - u_i\|_{L_2(\Omega)}$ by a computable estimator $\eta_{\text{pol}} = (\sum_{T \in \mathcal{T}_i} \eta_{\text{pol}}(T)^2)^{1/2}$, a task which can itself be rather technical (cf. [17]).

Next we list the key ingredients of typical convergence proofs of AFEM for controlling global energy errors. Based on the recent work [7], these are:

1. *An a posteriori error estimate.* A standard global estimate is given by (1.5) with $D = D_d = \Omega$ and the second term on the right hand side omitted.

2. *Orthogonality.*

$$\| \|u - u_{i+1}\| \|_{\Omega}^2 = \| \|u - u_i\| \|_{\Omega}^2 - \| \|u_i - u_{i+1}\| \|_{\Omega}^2. \quad (1.6)$$

3. *Estimator reduction.* Let $\tilde{\mathcal{M}}_i \subset \mathcal{T}_i$ be the set of elements marked for refinement in the i -th iteration of (1.2), and let $0 < \delta < 1$. There exists $0 < \lambda < 1$ depending on the refinement algorithm such that

$$\begin{aligned} \sum_{T \in \mathcal{T}_{i+1}} \eta_{1,i+1}(T)^2 &\leq (1 + \delta) \left(\sum_{T \in \mathcal{T}_i} \eta_{1,i}(T)^2 - \lambda \sum_{T \in \tilde{\mathcal{M}}_i} \eta_{1,i}(T)^2 \right) \\ &\quad + \frac{C}{\delta} \| \|u_i - u_{i+1}\| \|_{\Omega}^2. \end{aligned} \quad (1.7)$$

4. *Error contraction.* If $\tilde{\mathcal{M}}_i$ is appropriately chosen, there is stepwise reduction of error plus estimator: There exist $\gamma > 0$ and $0 < \alpha < 1$ such that

$$\| \|u - u_{i+1}\| \|_{\Omega}^2 + \gamma \sum_{T \in \mathcal{T}_{i+1}} \eta_{1,i+1}(T)^2 \leq \alpha^2 (\| \|u - u_i\| \|_{\Omega}^2 + \gamma \sum_{T \in \mathcal{T}_i} \eta_{1,i}(T)^2). \quad (1.8)$$

A chief challenge associated with proving convergence of an AFEM for controlling local energy errors is defining an error notion which admits orthogonality and estimator reduction properties. The local a posteriori estimate (1.5) exhibits unacceptable “spread” of the local energy error for purposes of proving convergence: The local energy norm on the left hand side is taken over D , but the energy residual terms on the right hand side are summed over D_d . This imbalance makes it difficult to contemplate proving meaningful local analogs of (1.6), (1.7), and (1.8) based on (1.5).

A main contribution of this work is the definition of a suitable local error notion which does not exhibit “error spread” as in (1.5), and for which we obtain reliable a posteriori upper bounds, an estimator reduction estimate, and contraction of the local energy error—but all modulo pollution terms measured in an L_p norm. In order to introduce our error notion, we note that the local estimates (1.4) and (1.5) are proved by introducing a smooth cutoff function ϕ which is 1 on D and 0 outside of D_d . ϕ typically is used only as an analytical aid, but it is not difficult to prove using the methods of [23] and [10] that in addition to (1.4), we have the sharper estimate

$$\| \|\phi(u - u_i)\| \|_{D_d} \leq C \min_{\chi \in S_i} \| \|\phi u - \chi\| \|_{D_d} + \frac{C}{d} \| \|u - u_i\| \|_{L_2(D_d)}. \quad (1.9)$$

Following (1.9), we choose $\| \|\phi(u - u_i)\| \|_{D_d}$ as the “goal quantity” for our adaptive algorithm. Under suitable restrictions on D , D_d , ϕ , and the marking scheme in the “mark” step in 1.2, we shall establish that modified versions of orthogonality, estimator reduction, and error contraction properties hold for a suitably defined adaptive algorithm for controlling $\| \|\phi(u - u_i)\| \|_{D_d}$. For $T \in \mathcal{T}_i$, we define the *weighted* energy residual indicator $\phi_{1,i,\phi}(T)^2 = h_T^2 \| \|\phi(f + \Delta u_i)\| \|_{L_2(\phi)}^2 + h_T \| \|\phi[\nabla u_i]\| \|_{L_2(\partial T)}^2$. Fixing $p = 2$ if Ω is convex and $4 < p < 6$ otherwise, define $M_i = C(d)[\eta_{0,i,p}(\mathcal{T}_i)^2 + \eta_{0,i+1,p}(\mathcal{T}_i)^2]$, where $\eta_{0,i,p}(\mathcal{T}_i)$ is an estimator for $\| \|u - u_i\| \|_{L_p(\Omega)}$ that we define later. Then we have:

1. *An a posteriori error estimate.*

$$\| \|\phi(u - u_i)\| \|_{D_d}^2 \leq C \sum_{T \cap D_d \neq \emptyset} \eta_{1,i,\phi}(T)^2 + C(d) \eta_{0,i,p}^2. \quad (1.10)$$

2. *Quasi-orthogonality, modulo M_i .* For any $\Lambda > 1$,

$$\begin{aligned} & |||\phi(u - u_{i+1})|||_{D_d}^2 \\ & \leq \Lambda |||\phi(u - u_i)|||_{D_d}^2 - |||\phi(u_i - u_{i+1})|||_{D_d}^2 + \frac{1}{\Lambda - 1} M_i. \end{aligned}$$

3. *Estimator reduction, modulo M_i .* Let \mathcal{M}_i be the set of elements marked for refinement in the i -th iteration of (1.2). Let $0 < \delta < 1$. With λ as in (1.7),

$$\begin{aligned} \sum_{T \subset D_d} \eta_{1,i+1,\phi}(T)^2 & \leq (1 + \delta) \left(\sum_{T \subset T} \eta_{1,i,\phi}(T)^2 - \lambda \sum_{T \in \mathcal{M}_i} \eta_{1,i,\phi}(T)^2 \right) \\ & \quad + C \left[\left(1 + \frac{1}{\delta}\right) |||\phi(u_i - u_{i+1})|||_{D_d}^2 + M_i \right]. \end{aligned}$$

4. *Error contraction, modulo M_i .* Define next the local quasi-error $E_i = |||\phi(u - u_i)|||_{D_d}^2 + \gamma \sum_{T \cap D_d \neq \emptyset} \eta_{1,i,\phi}(T)^2$. There exist $\gamma > 0$ and $0 < \alpha < 1$ such that

$$E_{i+1} \leq \alpha^2 E_i + M_i. \quad (1.11)$$

We may sum up our results as follows:

Heuristic for local energy AFEM: *An AFEM for $|||\phi(u - u_i)|||_{D_d}$ behaves like a standard global energy AFEM up to pollution terms.*

We finally prove three different convergence results for AFEM for controlling $|||\phi(u - u_i)|||_{D_d}$. First we characterize the effects of marking using *only* the local energy indicators $\eta_{1,i,\phi}$, that is, using an AFEM that does not control pollution effects. In this case we show that the adaptive algorithm converges to a limiting function $u_\infty \in H_0^1(\Omega)$ where in general $u_\infty \neq u$ even on D_d , but where $-\Delta u_\infty = f$ in D_d . Thus local refinement produces in the limit a solution which locally satisfies the correct partial differential equation, but not generally the correct boundary value problem.

We also prove two convergence results for an AFEM which adaptively controls pollution by employing an alternating marking based on the local energy estimators $\eta_{1,i,\phi}$ and the L_p estimators $\eta_{0,i,p}$. The first is a plain convergence result that as in the recent work [21] does not contain a rate of convergence, though the marking strategy we consider here is not as general as that allowed in [21]. We also show that our alternating marking strategy yields a contraction under somewhat more restrictive assumptions. This contraction result requires that Ω be convex so that we may measure the pollution error in the L_2 norm, that the marking strategy enforces marking for the pollution error sufficiently often, and that the sequence of meshes $\{\mathcal{T}_i\}$ is “mildly graded” in a sense that is made precise below. The latter result represents a substantial step towards proving optimality of an AFEM for controlling local energy errors, and we hope to consider the question of optimality in future work.

An outline of the paper is as follows. In §2 we establish appropriate preliminaries, including a precise definition of our AFEM and the introduction of an important technical tool, a *local reconstruction* of the local energy error. In §3 we establish the four “convergence ingredients” outlined in (1.10) through (1.11). In §4 we state, and prove, and discuss the three convergence results outlined above. In §5 we briefly describe some simple computational experiments that illustrate and confirm our theoretical results.

2. Preliminaries.

2.1. Mesh, finite element space, interpolants, and superapproximation.

Let $\{\mathcal{T}_i\}$, $i \geq 0$, be a sequence of simplicial decompositions of Ω such that all elements in all meshes are uniformly shape-regular and conforming (edge-to-edge), and \mathcal{T}_{i+1} is obtained by bisecting elements in \mathcal{T}_i in a manner which we make precise later. If \mathcal{T}_0 is shape-regular, then it is possible to guarantee that \mathcal{T}_i is shape-regular for all $i > 0$ by applying for example the newest-node bisection algorithm (cf. [25]). If $\omega \subset \Omega$, we define the local mesh with respect to ω by $\mathcal{T}_{i,\omega} = \{T \in \mathcal{T}_i : T \cap \omega \neq \emptyset\}$. For $T \in \mathcal{T}_i$, let $h_T = |T|^{1/n}$. In the next section we also describe a compatibility requirement between our target subdomain D , its extension D_d , and the family $\{\mathcal{T}_i\}$ of meshes.

Next let $S_i \subset H_0^1(\Omega)$ be a space of continuous Lagrange finite element functions which are piecewise polynomials of some fixed degree k on \mathcal{T}_i . Note that $S_i \subset S_{i+1}$, since \mathcal{T}_{i+1} is a refinement of \mathcal{T}_i . We shall employ two standard finite element (quasi)-interpolants. The first is the Lagrange interpolant, which we denote by I_L . The following two properties of I_L are used in our proofs below.

LEMMA 2.1. *Let $T \in \mathcal{T}_i$ for some $i \geq 0$, and let $\phi \in \mathbb{P}_j(T)$ for some fixed $j \geq 0$, where \mathbb{P}_j denotes the polynomials of degree j . Assume also that*

$$|\phi|_{W_\infty^j(T)} \leq C_\phi d^{-j}, \quad 0 \leq j \leq k+1, \quad (2.1)$$

for some $d \geq h_T$. Then for all $\chi \in S_i$, there holds the superapproximation estimate

$$\begin{aligned} \|\phi\chi - I_L(\phi\chi)\|_{L_2(T)} + h_T \|\phi\chi - I_L(\phi\chi)\|_{H^1(T)} \\ \leq Ch_T^2 \left(\frac{1}{d^2} \|\chi\|_{L_2(T)} + \frac{1}{d} \|\nabla\chi\|_{L_2(T)} \right) \\ \leq C \frac{h_T}{d} \|\chi\|_{L_2(T)}. \end{aligned} \quad (2.2)$$

In addition, for any $T \in \mathcal{T}_i$,

$$\|I_L(\phi\chi)\|_{H^1(T)} \leq C \|\phi\chi\|_{H^1(T)}, \quad (2.3)$$

The constants C above depend on the constant C_ϕ in (2.1), the degree k of the finite element space and j of ϕ , and the shape regularity of \mathcal{T}_i .

Proof. The superapproximation estimate (2.2) may be obtained by applying an inverse inequality to a standard superapproximation estimate (e.g., to (2.1) of [10]) while noting that $\frac{h_T}{d} \leq C$. In order to prove (2.3), we note that $\phi\chi$ belongs to a standard Lagrange finite element space of degree $k+j$. (2.3) may then be obtained by applying inverse estimates and the L_∞ stability of the Lagrange interpolant. \square

We shall also employ the Scott-Zhang interpolant I_{SZ} , which is stable in H^1 (cf. [26]). Let D_i be a fixed subdomain of Ω that is a union of elements in \mathcal{T}_i . Then we may define I_{SZ} so that

$$v \in H_0^1(D_i) \Rightarrow I_{SZ}v \in H_0^1(D_i).$$

We do not list further properties of I_{SZ} here, as its application in the establishment of residual-type a posteriori upper bounds is now rather standard.

2.2. Subdomain and cutoff function. Next we describe our target subdomain and cutoff function. For simplicity, we require that D be the intersection of a box (rectangle if $n = 2$) with Ω . Let D_d be the intersection with Ω of the box whose sides are parallel to and lie a distance d outside of those of D . We assume for simplicity that $d \leq 1$. Thus $D \subset D_d$ and $\text{dist}(D, \partial D_d \setminus \partial\Omega)$ is equivalent to d . We may also

consider sets D and D_d composed of nested connected components of intersections of such boxes with Ω if for example D abuts the crack in a crack domain. Let also $D' = D_d \setminus D$ denote the “ring” around D .

We require that D and D_d be mesh subdomains for each $i \geq 0$, that is, for each $i \geq 0$, $\bar{D} = \cup_{T \in \mathcal{T}_{i,D}} \bar{T}$ for some subset $\mathcal{T}_{i,D}$ of \mathcal{T}_i , and likewise for D_d and D' . This is not difficult to achieve given the geometry of D , D_d , and Ω .

Next we define a cutoff function ϕ which will play an integral role in our development. We require that:

C1. $\phi \in W_\infty^2(D_d) \cap W_\infty^1(\Omega)$.

C2. $0 < \phi \leq 1$ on (the interior of) D_d , $\phi = 0$ on $\Omega \setminus D_d$, and $\phi \equiv 1$ on D .

C3. ϕ is a piecewise polynomial of some fixed degree j with respect to each mesh \mathcal{T}_i , $i \geq 0$.

C4. There holds the estimate

$$|\phi|_{W_\infty^i(D_d)} \leq C_\phi d^{-i}, \quad i = 0, 1, \dots, j. \quad (2.4)$$

The construction of ϕ satisfying C1-C4 is straightforward given the restricted geometry of D ; it suffices to take the tensor product of functions of one variable satisfying one-dimensional versions of C1-C4. Note also that (2.1) and (2.2) hold for ϕ satisfying C1-C4 since for $T \in D'$, $\frac{h_T}{d} \leq C$.

We finally note the slightly non-standard Poincaré inequality

$$\|v\|_{L_2(D')} \leq C_P d \|\nabla v\|_{L_2(D')}, \quad v \in H_0^1(D_d). \quad (2.5)$$

If D_d lies on the interior of Ω , (2.2) may be proved by breaking D' into boxes, each of which has diameter d and each of which has one side lying on ∂D_d , and applying a Poincaré inequality on each of the individual subboxes. We may do so because $v = 0$ on one side of each subbox. If D_d abuts $\partial\Omega$, we may extend v to be 0 outside of Ω and then employ the same procedure.

2.3. Error estimators and indicators. We shall employ several different kinds of local residual-type error indicators at various points in our presentation. As in the introduction, we define an H^1 residual indicator weighted by the cutoff function ϕ ,

$$\eta_{1,i,\phi}(v_i, T)^2 = h_T^2 \|\phi(f + \Delta v_i)\|_{L_2(T)}^2 + h_T \|\phi[\nabla v_i]\|_{L_2(\partial T)}^2, \quad T \in \mathcal{T}_i \text{ and } v_i \in S_i. \quad (2.6)$$

For $2 \leq p < \infty$ we define the L_p indicator

$$\eta_{0,i,p}(v_i, T)^p = h_T^{2p} \|f + \Delta v_i\|_{L_p(T)}^p + h_T^{p+1} \|[\nabla v_i]\|_{L_p(\partial T)}^p, \quad T \in \mathcal{T}_i \text{ and } v_i \in S_i. \quad (2.7)$$

We will omit v_i in the above notation when $v_i = u_i$. For example, $\eta_{1,i,\phi}(T) = \eta_{1,i,\phi}(u_i, T)$.

Next we define error estimators. If $\mathcal{T} \subset \mathcal{T}_i$, we define

$$\begin{aligned} \eta_{1,i,\phi}(v_i, \mathcal{T})^2 &= \sum_{T \in \mathcal{T}} \eta_{1,i,\phi}(v_i, T)^2, \\ \eta_{0,i,p}(v_i, \mathcal{T})^p &= \sum_{T \in \mathcal{T}} \eta_{0,i,p}(v_i, T)^p, \end{aligned}$$

If $v_i = u_i$, we shall omit the reference to v_i in our notation as above.

2.4. An a posteriori error estimate for $\|u - u_i\|_{L_p(\Omega)}$. We shall measure the pollution error in an L_p norm, and thus we must establish that $\|u - u_i\|_{L_p(\Omega)} \leq C\eta_{0,i,p}(\mathcal{T}_i)$ for appropriately chosen p .

LEMMA 2.2. *Let $p = 2$ if Ω is convex and $4 < p < \infty$ otherwise. Then there exists C_p depending on p such that*

$$\|u - u_i\|_{L_p(\Omega)} \leq C_p \eta_{0,i,p}(\mathcal{T}_i). \quad (2.8)$$

Proof. (2.8) is standard if $p = 2$ and Ω is convex (cf. [17]). In the general case, (2.8) was observed in [9]. The proof relies in a straightforward manner on W_q^2 regularity results for polyhedral domains; cf. [8]. \square

2.5. Constants. It is important to establish that constants appearing in our results below do not depend on essential quantities except as explicitly stated. We shall employ generic constants C and C_i , $i = 0, 1, 2, \dots$ which depend on the shape-regularity of the family $\{\mathcal{T}_i\}_{i \geq 0}$ of meshes; the constant C_ϕ appearing in (2.4); the domain Ω and space dimension n ; the polynomial degrees k of the finite element spaces S_i and j of the cutoff function ϕ ; and the constant C_P in the Poincaré inequality (2.5). C and C_i may also depend on the regularity constant C_p in (2.8) in the case $p = 2$, but not otherwise.

We also employ a constant C_p which like C_p in (2.8) depends on p as well as the above quantities. The dependence of C_p upon p is difficult to establish in general, but in the case of smooth domains the p -dependence arises in a $W_q^2(\Omega)$ regularity result from the Marcinkiewicz interpolation theorem and is bounded by C_p . The range of values p values allowed in (2.8) and the corresponding constants also depend on the regularity of the elliptic coefficient matrix if it differs from the identity. In particular, it may be necessary to choose p to be larger than above if a piecewise-constant coefficient matrix is considered. Constants which depend upon quantities other than the above will be defined as necessary.

2.6. Local reconstruction and Galerkin orthogonality. We next introduce a local reconstruction which will play an important role in our analysis. The local reconstruction may be viewed as an a posteriori dual to the *local finite element projection* used in the classical work [23] of Nitsche and Schatz to prove local a priori error estimates, and is similar in spirit to the elliptic reconstruction used in [18] to obtain a posteriori estimates for parabolic problems. Given a finite element solution $u_i \in S_i$, we let $\mathcal{R}^i \in H_0^1(D_d)$ solve

$$-\Delta \mathcal{R}^i = \phi f - 2\nabla \phi \nabla u_i - u_i \Delta \phi. \quad (2.9)$$

Formally we have $-\Delta \mathcal{R}^i = -\Delta(\phi u_i) + \phi \Delta u_i - \Delta \phi u_i$. Thus \mathcal{R}^i solves a *continuous* local elliptic problem whose data lies between $-\Delta(\phi u)$ and $-\Delta(\phi u_i)$.

We next estimate $\|\mathcal{R}^i - \phi u\|_{D_d}$ and $\|\mathcal{R}^i - \mathcal{R}^{i+1}\|_{D_d}$.

LEMMA 2.3. *For any $2 \leq p < \infty$,*

$$\|\mathcal{R}^i - \phi u\|_{D_d} \leq C d^{-\frac{1}{2} - \frac{1}{p}} \|u - u_i\|_{L_p(D')} \quad (2.10)$$

and

$$\|\mathcal{R}^i - \mathcal{R}^{i+1}\|_{D_d} \leq C d^{-\frac{1}{2} - \frac{1}{p}} \|u_i - u_{i+1}\|_{L_p(D')}, \quad (2.11)$$

where C is the generic constant defined in §2.5.

Proof. Letting (\cdot, \cdot) represent the L_2 inner product, employing the weak forms of (1.1) and (2.9), noting that $\nabla\phi$ and $\Delta\phi$ are identically 0 on D , and integrating by parts, we calculate

$$\begin{aligned}
& \| \mathcal{R}^i - \phi u \|_{D_d}^2 \\
&= [A(\mathcal{R}^i, \mathcal{R}^i - \phi u)] - [(\phi \nabla u, \nabla(\mathcal{R}^i - \phi u))] - [(u \nabla \phi, \nabla(\mathcal{R}^i - \phi u))] \\
&= [(\phi f - 2\nabla\phi \nabla u_i - u_i \Delta\phi, \mathcal{R}^i - \phi u)] - [(\nabla u, \nabla[\phi(\mathcal{R}^i - \phi u)]) \\
&\quad - (\nabla u \nabla \phi, \mathcal{R}^i - \phi u)] + [(\nabla u \nabla \phi, \mathcal{R}^i - \phi u) + (u \Delta\phi, \mathcal{R}^i - \phi u)] \\
&= (\phi f + 2\nabla\phi \nabla(u - u_i) + (u - u_i)\Delta\phi, \mathcal{R}^i - \phi u) - (f, \phi(\mathcal{R}^i - \phi u)) \\
&= -((u - u_i)\Delta\phi, \mathcal{R}^i - \phi u) - 2(u - u_i, \nabla\phi \nabla(\mathcal{R}^i - \phi u)) \\
&\leq \|u - u_i\|_{L_2(D')} \left(\frac{C}{d^2} \|\mathcal{R}^i - \phi u\|_{L_2(D')} + \frac{1}{d} \| \mathcal{R}^i - \phi u \|_{D'} \right).
\end{aligned}$$

Applying (2.5) to the term $\|\mathcal{R}^i - \phi u\|_{L_2(D')}$ and dividing through by $\| \mathcal{R}^i - \phi u \|_{D_d}$ completes the proof of (2.10) in the case $p = 2$. Applying Hölder's inequality while noting that $\text{vol}(D') \leq Cd$ completes the proof of (2.10) for $2 < p < \infty$. The proof of (2.11) is similar. \square

We next state a perturbed Galerkin orthogonality result.

LEMMA 2.4. *Assume that $\chi \in S_i \cap H_0^1(D_d)$. Then for $2 \leq p < \infty$,*

$$\begin{aligned}
A(\mathcal{R}^i - \phi u_i, \chi) &= (f, \phi\chi - I_L(\phi\chi)) - A(u_i, \phi\chi - I_L(\phi\chi)) \\
&\leq Cd^{-\frac{1}{2}-\frac{1}{p}} \eta_{0,i,p}(\mathcal{T}_{i,D'}) \| \chi \|_{D_d}.
\end{aligned} \tag{2.12}$$

Proof. Integration by parts yield for $\chi \in S_i \cap H_0^1(D_d)$ that

$$\begin{aligned}
A(\phi u_i, \chi) &= A(u_i, \phi\chi) + (u_i, \nabla\phi \nabla\chi) - (\nabla u_i \nabla\phi, \chi) \\
&= A(u_i, \phi\chi - I_L(\phi\chi)) + (f, I_L(\phi\chi)) - (u_i \Delta\phi, \chi) - 2(\nabla u_i \nabla\phi, \chi).
\end{aligned}$$

Applying the weak form of (2.9), we then find that

$$A(\mathcal{R}^i - \phi u_i, \chi) = (f, \phi\chi - I_L(\phi\chi)) - A(u_i, \phi\chi - I_L(\phi\chi)),$$

which is the first equality in (2.12). In order to prove the inequality in (2.12), we use (2.2) along with standard techniques for manipulating residual-type error indicators while noting that $\phi\chi - I_L(\phi\chi) = 0$ on $\Omega \setminus D'$ (including on $\partial D'$) to compute

$$\begin{aligned}
& (f, \phi\chi - I_L(\phi\chi)) - A(u_i, \phi\chi - I_L(\phi\chi)) \\
&= \sum_{T \in \mathcal{T}_{i,D'}} \int_T (f + \Delta u_i)(\phi\chi - I_L(\phi\chi)) \, dx + \frac{1}{2} \int_{\partial T} \llbracket \nabla u_i \rrbracket (\phi\chi - I_L(\phi\chi)) \, d\sigma \\
&\leq \frac{C}{d} \sum_{T \in \mathcal{T}_{i,D'}} \eta_{0,i,2}(T) \left(\frac{1}{d} \|\chi\|_{L_2(T)} + \| \chi \|_T \right) \\
&\leq \frac{C}{d} \left(\sum_{T \in \mathcal{T}_{i,D'}} \eta_{0,i,2}(T)^2 \right)^{1/2} \left(\frac{1}{d} \|\chi\|_{L_2(D')} + \| \chi \|_{D_d} \right).
\end{aligned}$$

The desired result (2.12) is obtained for $p = 2$ by applying the Poincaré inequality (2.5) to the term $\frac{1}{d} \|\chi\|_{L_2(D')}$. The result for $p < 2 < \infty$ may be obtained by applying Hölder's inequality to the residual term. \square

Remark 2.5. Our results below may be proved without employing the local reconstruction, just as local a priori error estimates may be proved without using local finite element projections (cf. [10]). However, the local reconstruction clearly evokes the strategy of splitting the overall local error $|||\phi(u - u_i)|||_{D_d}$ into a local energy term which may be bounded a posteriori, plus a global pollution term measured in a weaker norm. In addition, we will obtain a posteriori estimates for $|||\mathcal{R}^i - \phi u_i|||_{D_d}$ that are in a sense stronger than those for $|||\phi(u - u_i)|||_{D_d}$.

2.7. Adaptive FEM. In this subsection we give details of our adaptive FEM. In particular, we give precise definitions of each module `solve`, `estimate`, `mark`, and `refine` of the generic adaptive iteration (1.2), with the goal of constructing an AFEM for reducing the weighted energy error $|||\phi(u - u_i)|||_{D_d}$. We assume that an initial shape-regular simplicial mesh \mathcal{T}_0 is given.

1. Module `solve`. Given a discrete space S_i defined on \mathcal{T}_i , solve (1.3) for u_i . We assume that the finite element system is assembled and solved exactly.
2. Module `estimate`. Let $2 \leq p < \infty$ be such that $\|u - u_i\|_{L_p(\Omega)} \leq C\eta_{0,i,p}(\mathcal{T}_i)$ as in Lemma 2.2. We prove below that the local error $|||\phi(u - u_i)|||_{D_d}$ is then bounded by

$$\mathcal{E}_{loc} = C\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + C_p d^{-\frac{1}{2} - \frac{1}{p}} \eta_{0,i,p}(\mathcal{T}_i). \quad (2.13)$$

In principle, the adaptive algorithm is terminated when $\mathcal{E}_{loc} \leq tol$ for some prescribed tolerance tol .

3. Module `mark`. Our basic marking strategy is a Dörfler marking (cf. [12]) to control the local error. Fixing $0 < \theta < 1$, at each step of the adaptive algorithm we determine $\mathcal{M}_i \subset \mathcal{T}_i$ so that

$$\eta_{1,i,\phi}(\mathcal{M}_i)^2 \geq \theta^2 \eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2. \quad (2.14)$$

The marking (2.14) will be used in §3 in order to prove a “contraction-modulo-pollution” estimate. In §4, we will modify (2.14) in order to obtain various convergence results.

4. Module `refine`. Our results below assume that each marked element $T \in \mathcal{M}_i$ is bisected $b \geq 1$ times in passing from \mathcal{T}_i to \mathcal{T}_{i+1} and that additional elements are refined in the process in order to ensure that \mathcal{T}_{i+1} is conforming. The newest-node algorithm employed for example by the finite element toolbox ALBERTA ([25]) enforces these conditions.

2.8. An abstract convergence result for AFEM. The following abstract lemma will play an important role in obtaining plain convergence results (without rates of convergence) and characterizing the limiting functions in our adaptive algorithms. We employ Lemma 6.1 of [2]. A more general version is given in [22], where it plays a fundamental role in establishing plain convergence results for AFEM.

LEMMA 2.6. *Let H be a Hilbert space and let $S_1 \subseteq S_2 \subseteq \dots \subseteq S_i \subseteq S_{i+1} \subseteq \dots \subseteq H$ be a nested sequence of closed subspaces. Let P_i denote the orthogonal projection onto S_i and let P_∞ denote the orthogonal projection onto $S_\infty = \bigcup_{i=1}^\infty S_i$. Then for any $u \in H$,*

$$P_i u \rightarrow P_\infty u \text{ as } i \rightarrow \infty. \quad (2.15)$$

3. Ingredients for convergence. In this section we prove an a posteriori error estimate, a quasi-orthogonality property, an estimator reduction inequality, and error contraction and stability properties, but all modulo terms which are roughly equivalent with global L_p norms of the error and which measure “pollution effects” of global solution properties upon local solution quality. Note also that the statement of results and proofs in this section require only that u and u_i solve (1.1) and (1.3) *locally on D_d* . Thus u may solve a PDE which could for example be degenerate elliptic or change type to hyperbolic outside of D_d .

3.1. A posteriori error estimate. In this section we establish a posteriori upper bounds for $|||\mathcal{R}^i - \phi u_i|||_{D_d}$ and $|||\phi(u - u_i)|||_{D_d}$.

THEOREM 3.1. *There exists a positive constant C_1 as defined in §2.5 such that for $2 \leq p < \infty$,*

$$|||\mathcal{R}^i - \phi u_i|||_{D_d}^2 \leq C_1 [\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 + d^{-1-\frac{2}{p}} \eta_{0,i,p}(\mathcal{T}_{i,D'})^2]. \quad (3.1)$$

Proof. Noting that $|||\mathcal{R}^i - \phi u_i|||_{D_d} = \sup_{\psi \in H_0^1(D_d), |||\psi|||_{D_d}=1} A(\mathcal{R}^i - \phi u_i, \psi)$, we let $\psi \in H_0^1(D_d)$ with $|||\psi|||_{D_d} = 1$. Using (2.12) and standard techniques for proving residual-type a posteriori bounds while noting that $-\Delta \mathcal{R}^i + \Delta(\phi u_i) = \phi(f + \Delta u_i)$ on each $T \in \mathcal{T}_i$, we then compute

$$\begin{aligned} A(\mathcal{R}^i - \phi u_i, \psi) &= A(\mathcal{R}^i - \phi u_i, \psi - I_{SZ}\psi) \\ &\quad + (f, \phi I_{SZ}\psi - I_L(\phi I_{SZ}\psi)) - A(u_i, \phi I_{SZ}\psi - I_L(\phi I_{SZ}\psi)) \\ &= \sum_{T \subset D_d} \int_T (-\Delta \mathcal{R}^i + \Delta(\phi u_i))(\psi - I_{SZ}\psi) \, dx + \frac{1}{2} \int_{\partial T} \llbracket \nabla(\phi u_i) \rrbracket (\psi - I_{SZ}\psi) \, d\sigma \\ &\quad + (f, \phi I_{SZ}\psi - I_L(\phi I_{SZ}\psi)) - A(u_i, \phi I_{SZ}\psi - I_L(\phi I_{SZ}\psi)) \\ &\leq C \sum_{T \subset D_d} (h_T \|\phi(f + \Delta u_i)\|_{L_2(T)} + h_T^{1/2} \|\llbracket \nabla(\phi u_i) \rrbracket\|_{L_2(\partial T \setminus \partial D_d)}) |||\psi|||_{D_d} \\ &\quad + C d^{-1} \eta_{0,i,2}(\mathcal{T}_{i,D'}) |||I_{SZ}\psi|||_{D_d}. \end{aligned} \quad (3.2)$$

We next note that $\|\llbracket \nabla(\phi u_i) \rrbracket\|_{L_2(\partial T \setminus \partial D_d)} = \|\llbracket \nabla u_i \rrbracket\|_{L_2(\partial T)}$ since $\phi \in C^1(D_d)$ by assumption C1. By the H^1 stability of I_{SZ} , $|||I_{SZ}\psi|||_{D_d} \leq C |||\psi|||_{D_d} \leq C$. Inserting this equation and estimate into (3.2), squaring the result, and setting $C_1 = C$ completes the proof (3.1) for $p = 2$. Applying Hölder’s inequality to the L_2 -type residual term above yields the final result. \square

Applying the triangle and Young’s inequalities to (3.1) while recalling (2.10) and Lemma 2.2 trivially yields the following corollary.

COROLLARY 3.2. *For any $2 \leq p < \infty$,*

$$|||\phi(u - u_i)|||_{D_d}^2 \leq C_2 [\eta_{1,i,\phi}(\mathcal{T}_{D_d})^2 + d^{-1-\frac{2}{p}} (\eta_{0,i,p}(\mathcal{T}_{D'})^2 + \|u - u_i\|_{L_p(D')}^2)], \quad (3.3)$$

where C_2 is defined in §2.5. Thus for $p = 2$ if Ω is convex and $4 < p < \infty$ otherwise,

$$|||\phi(u - u_i)|||_{D_d}^2 \leq C_2 \eta_{1,i,\phi}(\mathcal{T}_{D_d})^2 + C_p d^{-1-\frac{2}{p}} \eta_{0,i,p}(\mathcal{T}_i)^2. \quad (3.4)$$

Remark 3.3. We emphasize the difference between Theorem 3.1 and Corollary 3.2. (3.1) provides an upper bound for the local reconstruction error $|||\mathcal{R}^i - \phi u_i|||_{D_d}$ which

depends on u_i and f only in the set D_d . The proof of this result does not rely upon a duality argument, as the L_2 -type error estimators in the upper bound arise from a superapproximation estimate. On the other hand, the bound (3.3) for the overall local error $|||\phi(u-u_i)|||_{D_d}$ repeats the error $u-u_i$ on the right-hand-side (albeit in a weaker norm), and bounding the resulting pollution term $\|u-u_i\|_{L_p(D')}$ a posteriori requires the use of a duality argument as in Lemma 2.2. Thus the reconstruction \mathcal{R}^i provides a splitting of $|||\phi(u-u_i)|||_{D_d}$ into a “local energy error” $|||\mathcal{R}^i - \phi u_i|||_{D_d}$ which may be bounded a posteriori using information only from the set D_d , and a “pollution term” $|||\mathcal{R}^i - \phi u|||_{D_d} \leq Cd^{-1}\|u-u_i\|_{L_2(D')}$ which measures global influences and which cannot be bounded a posteriori using information only from D_d .

3.2. Quasi-orthogonality. We next prove a *quasi-orthogonality* result (cf. [19] for a similar estimate in the context of convergence of AFEM in the global energy norm for general second-order linear elliptic problems).

THEOREM 3.4. *For any $1 < \Lambda_1 \leq 2$ and $p \leq 2 < \infty$,*

$$\begin{aligned} |||\mathcal{R}^{i+1} - \phi u_{i+1}|||_{D_d}^2 &\leq \Lambda_1 |||\mathcal{R}^i - \phi u_i|||_{D_d}^2 - |||\phi(u_i - u_{i+1})|||_{D_d}^2 \\ &\quad + \frac{C}{(\Lambda_1 - 1)} d^{-1-\frac{2}{p}} (\|u_i - u_{i+1}\|_{L_p(D')}^2 + \eta_{0,i+1,p}(\mathcal{T}_{i+1,D'})^2). \end{aligned} \quad (3.5)$$

Here C is the generic constant defined in §2.5.

Proof. We first calculate

$$\begin{aligned} |||\mathcal{R}^i - \phi u_i|||_{D_d}^2 &= |||(\mathcal{R}^i - \mathcal{R}^{i+1}) + (\mathcal{R}^{i+1} - \phi u_{i+1}) + \phi(u_{i+1} - u_i)|||_{D_d}^2 \\ &= |||\mathcal{R}^{i+1} - \phi u_{i+1}|||_{D_d}^2 + |||\phi(u_i - u_{i+1})|||_{D_d}^2 \\ &\quad + A(\mathcal{R}^i - \mathcal{R}^{i+1}, \mathcal{R}^i - \phi u_i + \mathcal{R}^{i+1} - \phi u_{i+1} + \phi(u_{i+1} - u_i)) \\ &\quad + 2A(\mathcal{R}^{i+1} - \phi u_{i+1}, \phi(u_{i+1} - u_i)). \end{aligned} \quad (3.6)$$

Next we employ (2.11) along with Young’s inequality to compute that for any $\epsilon > 0$

$$\begin{aligned} &A(\mathcal{R}^i - \mathcal{R}^{i+1}, \mathcal{R}^i - \phi u_i + \mathcal{R}^{i+1} - \phi u_{i+1} + \phi(u_{i+1} - u_i)) \\ &= -A(\mathcal{R}^i - \mathcal{R}^{i+1}, \mathcal{R}^i - \mathcal{R}^{i+1}) + 2(\mathcal{R}^i - \mathcal{R}^{i+1}, \mathcal{R}^i - \phi u_i) \\ &\leq (1 + \frac{1}{\epsilon}) |||\mathcal{R}^i - \mathcal{R}^{i+1}|||_{D_d}^2 + \epsilon |||\mathcal{R}^i - \phi u_i|||_{D_d}^2 \\ &\leq C(1 + \frac{1}{\epsilon}) d^{-1-\frac{2}{p}} \|u_i - u_{i+1}\|_{L_p(D')}^2 + \epsilon |||\mathcal{R}^i - \phi u_i|||_{D_d}^2 \end{aligned} \quad (3.7)$$

Letting I_L now be the Lagrange interpolant into S_{i+1} , we use (2.12) and (2.3) along with Young’s inequality to compute that for $\epsilon > 0$ as above,

$$\begin{aligned} &2A(\mathcal{R}^{i+1} - \phi u_{i+1}, \phi(u_{i+1} - u_i)) \\ &\leq 2|A(\mathcal{R}^{i+1} - \phi u_{i+1}, \phi(u_{i+1} - u_i) - I_L(\phi(u_{i+1} - u_i)))| \\ &\quad + Cd^{-\frac{1}{2}-\frac{1}{p}} \eta_{0,i+1,p}(\mathcal{T}_{i+1,D'}) |||I_L(\phi(u_{i+1} - u_i))|||_{D_d} \\ &\leq 2|A(\mathcal{R}^{i+1} - \phi u_{i+1}, \phi(u_{i+1} - u_i) - I_L(\phi(u_{i+1} - u_i)))| \\ &\quad + \frac{C}{\epsilon} d^{-1-\frac{2}{p}} \eta_{0,i+1,p}(\mathcal{T}_{i+1,D'})^2 + \epsilon |||\phi(u_{i+1} - u_i)|||_{D_d}^2. \end{aligned} \quad (3.8)$$

Employing the superapproximation result (2.2) and Hölder's inequality, we have for $\epsilon > 0$ as above that

$$\begin{aligned}
& 2A(\mathcal{R}^{i+1} - \phi u_{i+1}, \phi(u_{i+1} - u_i) - I_L(\phi(u_{i+1} - u_i))) \\
& \leq \sum_{T \in \mathcal{T}_{i+1, D'}} Cd^{-1} \|\mathcal{R}^{i+1} - \phi u_{i+1}\|_T \|u_{i+1} - u_i\|_{L_2(T)} \\
& \leq \frac{C}{\epsilon} d^{-1-\frac{2}{p}} \|u_{i+1} - u_i\|_{L_p(D')}^2 + \epsilon \|\mathcal{R}^{i+1} - \phi u_{i+1}\|_{D_d}^2.
\end{aligned} \tag{3.9}$$

Noting from (3.6) that

$$\begin{aligned}
\|\mathcal{R}^{i+1} - \phi u_{i+1}\|_{D_d}^2 &= \|\mathcal{R}^i - \phi u_i\|_{D_d}^2 - \|\phi(u_i - u_{i+1})\|_{D_d}^2 \\
&\quad - A(\mathcal{R}^i - \mathcal{R}^{i+1}, \mathcal{R}^i - \phi u_i + \mathcal{R}^{i+1} - \phi u_{i+1} + \phi(u_{i+1} - u_i)) \\
&\quad - 2A(\mathcal{R}^{i+1} - \phi u_{i+1}, \phi(u_{i+1} - u_i)),
\end{aligned}$$

we combine (3.7), (3.8), and (3.9) to obtain for any $\epsilon > 0$ that

$$\begin{aligned}
(1 - \epsilon) \|\mathcal{R}^{i+1} - \phi u_{i+1}\|_{D_d}^2 &\leq (1 + \epsilon) \|\mathcal{R}^i - \phi u_i\|_{D_d}^2 - (1 - \epsilon) \|\phi(u_i - u_{i+1})\|_{D_d}^2 \\
&\quad + C(1 + \frac{1}{\epsilon}) d^{-1-\frac{2}{p}} (\|u_i - u_{i+1}\|_{L_p(D')}^2 + \eta_{0, i+1, p}(\mathcal{T}_{i+1, D'})^2).
\end{aligned} \tag{3.10}$$

Dividing (3.10) through by $1 - \epsilon$ and setting $\epsilon = \frac{\Lambda_1 - 1}{\Lambda_1 + 1}$, we obtain (3.5). \square

We finally prove a quasi-orthogonality result for $\|\phi(u - u_i)\|_{D_d}$.

COROLLARY 3.5. *For any $1 < \Lambda_2 \leq 2$ and any fixed $2 \leq p < \infty$,*

$$\begin{aligned}
\|\phi(u - u_{i+1})\|_{D_d}^2 &\leq \Lambda_2 \|\phi(u - u_i)\|_{D_d}^2 - \|\phi(u_i - u_{i+1})\|_{D_d}^2 \\
&\quad + C \frac{1}{\Lambda_2 - 1} d^{-1-\frac{2}{p}} (\|u - u_i\|_{L_p(D')}^2 + \|u - u_{i+1}\|_{L_p(D')}^2 + \eta_{0, i+1, p}(\mathcal{T}_{i+1, D'})^2).
\end{aligned} \tag{3.11}$$

Here C is the generic constant defined in §2.5.

Proof. Employing (2.10) and (3.5) along with the triangle inequality and Young's inequality, we compute that for any $0 < \delta \leq 1$ and $1 < \Lambda_1 \leq 2$,

$$\begin{aligned}
\|\phi(u - u_{i+1})\|_{D_d}^2 &\leq (1 + \delta) \|\mathcal{R}^{i+1} - \phi u_{i+1}\|_{D_d}^2 + (1 + \frac{1}{\delta}) \|\mathcal{R}^{i+1} - \phi u\|_{D_d}^2 \\
&\leq \Lambda_1(1 + \delta) \|\mathcal{R}^i - \phi u_i\|_{D_d}^2 - (1 + \delta) \|\phi(u_i - u_{i+1})\|_{D_d}^2 \\
&\quad + Cd^{-1-\frac{2}{p}} (1 + \frac{1}{\delta} + \frac{1}{\Lambda_1 - 1}) [\|u - u_{i+1}\|_{L_p(D')}^2 + \|u - u_i\|_{L_p(D')}^2 \\
&\quad + \eta_{0, i+1, p}(\mathcal{T}_{i+1, D'})^2] \\
&\leq \Lambda_1(1 + \delta)^2 \|\phi(u - u_i)\|_{D_d}^2 + \Lambda_1(1 + \delta)(1 + \frac{1}{\delta}) \|\mathcal{R}^i - \phi u_i\|_{D_d}^2 \\
&\quad - \|\phi(u_i - u_{i+1})\|_{D_d}^2 \\
&\quad + Cd^{-1-\frac{2}{p}} (1 + \frac{1}{\delta} + \frac{1}{\Lambda_1 - 1}) [\|u - u_{i+1}\|_{L_p(D')}^2 + \|u - u_i\|_{L_p(D')}^2 \\
&\quad + \eta_{0, i+1, p}(\mathcal{T}_{i+1, D'})^2] \\
&\leq \Lambda_1(1 + \delta)^2 \|\phi(u - u_i)\|_{D_d}^2 - \|\phi(u_i - u_{i+1})\|_{D_d}^2 \\
&\quad + Cd^{-1-\frac{2}{p}} (1 + \frac{1}{\delta} + \frac{1}{\Lambda_1 - 1}) [\|u - u_{i+1}\|_{L_p(D')}^2 + \|u - u_i\|_{L_p(D')}^2 \\
&\quad + \eta_{0, i+1, p}(\mathcal{T}_{i+1, D'})^2].
\end{aligned}$$

Setting $\Lambda_1 = 1 + \delta$, choosing δ so that $(1 + \delta)^3 = \Lambda_2$, and noting that $\frac{1}{\delta} \leq C \frac{1}{(1+\delta)^3-1}$ completes the proof. \square

3.3. Estimator reduction. In this section we prove an estimator reduction lemma. Here we employ the techniques of [7], which do not directly rely upon local a posteriori lower bounds as do previous proofs of convergence of adaptive FEM. We first prove a simple residual perturbation result; cf. Proposition 3.3 of [7].

PROPOSITION 3.6. *Let \mathcal{T}_i be a mesh, $i \geq 0$. Then for any $T \in \mathcal{T}_{i,D_d}$ and any pair of discrete function $v_i, w_i \in S_i$*

$$\eta_{1,i,\phi}(v_i, T) \leq \eta_{1,i,\phi}(w_i, T) + C(\|\phi(v_i - w_i)\|_{\bar{\omega}_T} + d^{-1}\|v_i - w_i\|_{L_2(\omega_T \cap D')}), \quad (3.12)$$

Here $C > 0$ is defined in §2.5 and ω_T is the union of elements sharing a side with T and lying in \mathcal{T}_{i,D_d} .

Proof. Recalling the definition (2.6), we use the triangle inequality to compute for $T \in \mathcal{T}_{D_d}$ that

$$\begin{aligned} \eta_{1,i,\phi}(v_i, T) &\leq \eta_{1,i,\phi}(w_i, T) \\ &\quad + (h_T^2 \|\phi(\Delta v_i - \Delta w_i)\|_{L_2(T)}^2 + h_T \|\phi[\nabla(v_i - w_i)]\|_{L_2(\partial T)}^2)^{1/2}. \end{aligned} \quad (3.13)$$

Letting $\bar{\phi}_T = \|\phi\|_{L_\infty(T)}$, employing an inverse inequality, and noting that $\|\phi - \bar{\phi}_T\|_{L_\infty(T)} \leq \frac{Ch_T}{d}$, we compute

$$\begin{aligned} h_T \|\phi(\Delta v_i - \Delta w_i)\|_{L_2(T)} &\leq C\bar{\phi}_T \|v_i - w_i\|_T \\ &\leq C(\|\phi \nabla(v_i - w_i)\|_{L_2(T)} + \|\phi - \bar{\phi}_T\|_{L_\infty(T)} \|v_i - w_i\|_T) \\ &\leq C(\|\phi(v_i - w_i)\|_{L_2(T)} + \frac{C}{d} \|v_i - w_i\|_{L_2(T \cap D')}). \end{aligned} \quad (3.14)$$

Employing the trace inequality $\|\nabla v_i\|_{L_2(\partial T)} \leq C(h_T^{-1/2} \|\nabla v_i\|_{L_2(T)} + h_T^{1/2} |v_i|_{H^2(T)})$, we compute as above to obtain for the edge $e = T \cap T'$

$$\begin{aligned} h_T^{1/2} \|\phi[\nabla(v_i - w_i)]\|_{L_2(e)} &\leq \bar{\phi}_T h_T^{1/2} (\|\nabla(v_i - w_i)_T\|_{L_2(e)} + \|\nabla(v_i - w_i)_{T'}\|_{L_2(e)}) \\ &\leq C\bar{\phi}_T h_T^{1/2} (h_T^{-1/2} \|\nabla(v_i - w_i)\|_{L_2(T)} + h_T^{1/2} |v_i - w_i|_{H^2(T)} \\ &\quad + h_T^{-1/2} \|\nabla(v_i - w_i)\|_{L_2(T')} + h_T^{1/2} |v_i - w_i|_{H^2(T')}) \\ &\leq C(\|\phi(v_i - w_i)\|_{L_2(T \cup T')} + \|(v_i - w_i)\nabla\phi\|_{L_2(T \cup T')}). \end{aligned} \quad (3.15)$$

Above we have employed the convention that for x lying in the edge $e = T \cap T'$, $\nabla(v_i - w_i)_T(x) = \lim_{y \in T, y \rightarrow x} \nabla(v_i - w_i)(y)$. Noting that $\|(v_i - w_i)\nabla\phi\|_{L_2(T \cup T')} \leq Cd^{-1} \|v_i - w_i\|_{L_2(\omega_T \cap D')}$, summing over the edges of T lying in the interior of D_d and collecting (3.15) and (3.14) into (3.13) completes the proof of (3.12). \square

Next we establish that the AFEM leads to reduction of the estimator, modulo pollution terms. Our proof closely follows Corollary 3.4 of [7].

LEMMA 3.7. *Let \mathcal{T}_i , $i \geq 0$ be given, and let \mathcal{T}_{i+1} be obtained by bisecting $\mathcal{M}_i \subset \mathcal{T}_i$ at least b times. Let also $\lambda = 1 - 2^{-\frac{b}{n}}$. Then for any $v_i \in S_i$ and $v_{i+1} \in S_{i+1}$, any $0 < \delta \leq 1$, and any $2 \leq p < \infty$,*

$$\begin{aligned} \eta_{1,i+1,\phi}(v_{i+1}, \mathcal{T}_{i+1,D_d})^2 &\leq (1 + \delta)(\eta_{1,i,\phi}(v_i, D_d)^2 - \lambda \eta_{1,i,\phi}(v_i, \mathcal{M}_i)^2) \\ &\quad + C_3(1 + \frac{1}{\delta})[\|\phi(v_i - v_{i+1})\|_{D_d}^2 + d^{-1-\frac{2}{p}} \|v_i - v_{i+1}\|_{L_p(D')}^2]. \end{aligned} \quad (3.16)$$

Here C_3 is the generic constant defined in §2.5.

Proof. We first apply (3.12) to $v_i, v_{i+1} \in S_{i+1}$, square the result, and apply Young's inequality to the resulting mixed terms to obtain for any $0 < \delta \leq 1$ and $T \in \mathcal{T}_{i+1}$

$$\begin{aligned} \eta_{1,i+1,\phi}(v_{i+1}, T)^2 &\leq (1 + \delta)\eta_{1,i+1,\phi}(v_i, T)^2 \\ &\quad + C(2 + \frac{2}{\delta})[\|\phi(v_i - v_{i+1})\|_{\omega_T}^2 + d^{-2}\|v_i - v_{i+1}\|_{L_2(\omega_T \cap D')}^2]. \end{aligned}$$

Summing over $T \in \mathcal{T}_{i+1, D_d}$, using the fact that no element is contained in more than $n + 1$ patches ω_T , and applying Hölder's inequality, we have

$$\begin{aligned} \eta_{1,i+1,\phi}(v_{i+1}, \mathcal{T}_{i+1, D_d})^2 &\leq (1 + \delta)\eta_{1,i+1,\phi}(v_i, \mathcal{T}_{i+1, D_d})^2 \\ &\quad + C(1 + \frac{1}{\delta})[\|\phi(v_i - v_{i+1})\|_{D_d}^2 + d^{-1-\frac{2}{p}}\|v_i - v_{i+1}\|_{L_p(D')}^2]. \end{aligned} \quad (3.17)$$

Let now $\tilde{T} \in \mathcal{M}_i$ be a marked element. Note that $h_{T'} \leq 2^{-\frac{b}{n}}h_{\tilde{T}}$ for $T' \in \mathcal{T}_{i+1, \tilde{T}}$, and that $[\nabla v_i] = 0$ across interfaces of T' lying in the interior of \tilde{T} . Thus

$$\eta_{1,i+1,\phi}(v_i, \mathcal{T}_{i+1, \tilde{T}})^2 \leq 2^{-\frac{b}{n}}\eta_{1,i,\phi}(v_i, \tilde{T})^2.$$

Combining this with the trivially proved monotonicity property $\eta_{1,i+1,\phi}(v_i, T) \leq \eta_{1,i,\phi}(v_i, T)$ for $T \in \mathcal{T}_{i, D_d} \setminus \mathcal{M}_i$ and summing over $T \in \mathcal{T}_{i+1, D_d}$ yields

$$\begin{aligned} \eta_{1,i+1,\phi}(v_i, \mathcal{T}_{i+1, D_d})^2 &\leq \eta_{1,i,\phi}(v_i, \mathcal{T}_{i, D_d} \setminus \mathcal{M}_i)^2 + 2^{-\frac{b}{n}}\eta_{1,i,\phi}(v_i, \mathcal{M}_i)^2 \\ &= \eta_{1,i,\phi}(v_i, \mathcal{T}_{i, D_d})^2 - \lambda\eta_{1,i,\phi}(v_i, \mathcal{M}_i)^2. \end{aligned} \quad (3.18)$$

Combining (3.18) and (3.17) yields (3.16). \square

3.4. Error contraction. We finally establish contraction properties for the quantities $\|\mathcal{R}^i - \phi u_i\|_{D_d}^2 + \tilde{\gamma}\eta_{1,i,\phi}(\mathcal{T}_{i, D_d})^2$ and $\|\phi(u - u_i)\|_{D_d}^2 + \gamma\eta_{1,i,\phi}(\mathcal{T}_{i, D_d})^2$.

THEOREM 3.8. *Assume that the marking strategy (2.14) is employed. Then there exist constants $\tilde{\gamma} > 0$ and $0 < \tilde{\alpha} < 1$ depending only on the generic constant C defined in §2.5, the parameter θ in (2.14), and the number of times b that each element in \mathcal{M}_i is bisected such that for any $2 \leq p < \infty$,*

$$\begin{aligned} \|\mathcal{R}^{i+1} - \phi u_{i+1}\|_{D_d}^2 + \tilde{\gamma}\eta_{1,i+1,\phi}(\mathcal{T}_{i+1, D_d})^2 \\ \leq \tilde{\alpha}^2(\|\mathcal{R}^i - \phi u_i\|_{D_d}^2 + \tilde{\gamma}\eta_{1,i,\phi}(\mathcal{T}_{i, D_d})^2) + \tilde{M}_{p,i}, \end{aligned} \quad (3.19)$$

where

$$\tilde{M}_{p,i} = Cd^{-1-\frac{2}{p}}(\|u_i - u_{i+1}\|_{L_p(D')}^2 + \eta_{0,i,p}(\mathcal{T}_{i, D'})^2 + \eta_{0,i+1,p}(\mathcal{T}_{i+1, D'})^2). \quad (3.20)$$

Proof. We will use the abbreviations $\eta_i^2 = \eta_{1,i,\phi}(\mathcal{T}_{i, D_d})^2$, $\eta_i(\mathcal{M}_i)^2 = \eta_{1,i,\phi}(\mathcal{M}_i)^2$, $e_i^2 = \|\mathcal{R}^i - \phi u_i\|_{D_d}^2$, and $E_i^2 = \|\phi(u_i - u_{i+1})\|_{D_d}^2$. Combining (3.5) and (3.16) yields for any $\Lambda_1 > 1$ and $\delta > 0$

$$\begin{aligned} e_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 &\leq \Lambda_1 e_i^2 - E_i^2 + \tilde{\gamma}(1 + \delta)(\eta_i^2 - \lambda\eta_i(\mathcal{M}_i)^2) + \tilde{\gamma}C_3(1 + \frac{1}{\delta})E_i^2 \\ &\quad + (\frac{1}{\Lambda_1 - 1} + 1 + \frac{1}{\delta})\tilde{M}_{p,i}. \end{aligned} \quad (3.21)$$

Note that we are free to choose $\tilde{\gamma} > 0$ and $0 < \delta \leq 1$. With δ to be chosen later, we fix $\tilde{\gamma} = \tilde{\gamma}(\delta)$ so that

$$\tilde{\gamma} = \frac{\delta}{C_3(1+\delta)}, \quad \tilde{\gamma}(1+\frac{1}{\delta})C_3 = 1, \quad (3.22)$$

which when inserted into (3.21) yields

$$e_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 \leq \Lambda_1 e_i^2 + \tilde{\gamma}(1+\delta)(\eta_i^2 - \lambda\eta_i(\mathcal{M}_i)^2) + (\frac{1}{\Lambda_1-1} + 1 + \frac{1}{\delta})\tilde{M}_{p,i}. \quad (3.23)$$

Noting that $\mathcal{M}_{i,D_d} \subset \mathcal{M}_i$ and inserting (4.13) into (3.23), we calculate

$$e_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 \leq \Lambda_1 e_i^2 + \tilde{\gamma}(1+\delta)(1 - \lambda\theta^2)\eta_i^2 + (\frac{1}{\Lambda_1-1} + 1 + \frac{1}{\delta})\tilde{M}_{p,i}$$

Splitting the term $\tilde{\gamma}(1+\delta)\lambda\theta^2\eta_i^2$ into two equal parts, we thus have

$$\begin{aligned} e_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 &\leq \Lambda_1 e_i^2 + \tilde{\gamma}(1+\delta)(1 - \frac{1}{2}\lambda\theta^2)\eta_i^2 - \frac{1}{2}\tilde{\gamma}(1+\delta)\lambda\theta^2 e_i^2 \\ &\quad + (\frac{1}{\Lambda_1-1} + 1 + \frac{1}{\delta})\tilde{M}_{p,i} \end{aligned}$$

Next we apply (3.1) to the term $-\frac{1}{2}\tilde{\gamma}(1+\delta)\lambda\theta^2 e_i^2$ while recalling (3.22) to obtain

$$\begin{aligned} e_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 &\leq \Lambda_1 e_i^2 + \tilde{\gamma}(1+\delta)(1 - \frac{1}{2}\lambda\theta^2)\eta_i^2 \\ &\quad - \frac{1}{2}\tilde{\gamma}(1+\delta)\lambda\theta^2(\frac{1}{C_1}e_i^2 - \tilde{M}_{p,i}) + (\frac{1}{\Lambda_1-1} + 1 + \frac{1}{\delta})\tilde{M}_{p,i} \\ &\leq (\Lambda_1 - \frac{\delta\lambda\theta^2}{2C_1C_3})e_i^2 + \tilde{\gamma}(1+\delta)(1 - \frac{1}{2}\lambda\theta^2)\eta_i^2 \\ &\quad + (\frac{1}{\Lambda_1-1} + 1 + \frac{1}{\delta} + \frac{\delta\lambda\theta^2}{2C_1C_3})\tilde{M}_{p,i}. \end{aligned}$$

Choosing $\delta > 0$ sufficiently close to 0 to ensure that $(1+\delta)(1 - \frac{1}{2}\lambda\theta^2) < 1$, choosing $\Lambda_1 > 1$ sufficiently close to 1 to ensure that $\Lambda_1 - \frac{\delta\lambda\theta^2}{2C_1C_3} < 1$, and appropriately adjusting the constant in (3.20) yields (3.19) with

$$\tilde{\alpha}^2 = \max\{\Lambda_1 - \frac{\delta\lambda\theta^2}{2C_1C_3}, (1+\delta)(1 - \frac{1}{2}\lambda\theta^2)\}.$$

□

COROLLARY 3.9. *There exists constants $\gamma > 0$ and $0 < \alpha < 1$ depending only on the generic constant C defined in §2.5, the parameter θ in (2.14), and the number of times b that each element in \mathcal{M}_i is bisected such that for any $2 \leq p < \infty$,*

$$\begin{aligned} \|\phi(u - u_{i+1})\|_{D_d}^2 + \gamma\eta_{1,i+1,\phi}(\mathcal{T}_{i+1,D_d})^2 \\ \leq \alpha^2(\|\phi(u - u_i)\|_{D_d}^2 + \gamma\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2) + M_{p,i}, \end{aligned} \quad (3.24)$$

where

$$\begin{aligned} M_{p,i} &= C_4 d^{-1-\frac{2}{p}}(\|u - u_i\|_{L_p(D')}^2 + \|u - u_{i+1}\|_{L_p(D')}^2 \\ &\quad + \eta_{0,i,p}(\mathcal{T}_{i,D'})^2 + \eta_{0,i+1,p}(\mathcal{T}_{i+1,D'})^2). \end{aligned}$$

Proof. The proof of (3.24) proceeds as the proof of (3.19), except that (3.11) is used instead of (3.5) and (3.3) is used instead of (3.1). \square

Remark 3.10. While adaptive control of the pollution error is in principal desirable, the contraction result (3.24) is relevant to some practical situations considered in the literature. In [3] and [4], [27], and [16], adaptive parallel algorithms based on AFEM employing local energy estimators are proposed. These algorithms do not control pollution adaptively, but rather carry out adaptive calculations based only on local energy estimators. For example, [27] and [16] in essence propose to ensure (or assume) that the pollution error on the initial mesh \mathcal{T}_0 is controlled to within a given tolerance and does not grow on subsequent refined grids, then refine adaptively based on local energy estimators only. More precisely, let us assume that $M_{p,i} \leq \epsilon$ in (3.24) for all $i \geq 0$. Then one obtains from (3.24)

$$E_i^2 \leq \alpha^{2i} E_0^2 + \frac{1}{1 - \alpha^2} \epsilon. \quad (3.25)$$

Here $E_i = \|\phi(u - u_i)\|_{D_d}^2 + \gamma \eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2$. Thus (3.24) establishes that an AFEM based only on Dörfler marking for $\eta_{1,i,\phi}$ will contract until the local energy quasi-error E_i is approximately the same magnitude as the pollution error. It is likely that a quasi-optimality result analogous to (3.25) could also be established, though we do not pursue the details here.

4. Convergence results. In this section we state and prove three different convergence results for the adaptive algorithm defined in §2.7.

4.1. Estimator reduction for L_p estimators. We first prove an L_p counterpart of Lemma 3.7 (estimator reduction for weighted energy norms) that will be used in the proofs of two of the convergence results below.

LEMMA 4.1. *Let \mathcal{T}_i , $i \geq 0$ be given, and let \mathcal{T}_{i+1} be obtained by bisecting $\mathcal{M}_i \subset \mathcal{T}_i$ at least b times. Let also $\lambda_p = 1 - 2^{-\frac{(p+1)b}{n}}$. Then for any $v_i \in S_i$ and $v_{i+1} \in S_{i+1}$, any $0 < \delta \leq 1$, and any $2 \leq p < \infty$,*

$$\begin{aligned} \eta_{0,i+1,p}(v_{i+1}, \mathcal{T}_{i+1})^p &\leq (1 + \delta)(\eta_{0,i,p}(v_i, \mathcal{T}_i)^p - \lambda_p \eta_{0,i,p}(v_i, \mathcal{M}_i)^p) \\ &\quad + C_p \mu_p(\delta) \|v_i - v_{i+1}\|_{\Omega}^p. \end{aligned} \quad (4.1)$$

Here C_p is the generic constant defined in §2.5, and μ_p is a positive function depending on p such that $\mu_p(\delta) \rightarrow \infty$ as $\delta \rightarrow 0$ and where $\mu_2(\delta) = 1 + \frac{1}{\delta}$.

Proof. Analogous to Proposition 3.6, we have for $T \in \mathcal{T}_i$ and $v_i, w_i \in S_i$

$$\eta_{0,i,p}(v_i, T) \leq \eta_{0,i,p}(w_i, T) + C \|v_i - w_i\|_{L_p(\omega_T)}, \quad (4.2)$$

where ω_T is the patch of elements sharing a side with T . We omit the proof.

The chief difference between our proof and that of Lemma 3.7 is the following result. Suppose that $a, b, c \geq 0$ with $a \leq b + c$. Then there exists constants C_p and functions $\mu_p(\delta)$ as above such that for $\delta > 0$,

$$a^p \leq (1 + \delta)b^p + C_p \mu_p(\delta)c^p. \quad (4.3)$$

$\mu_2(\delta) = 1 + \frac{1}{\delta}$ since $2bc \leq \delta b^2 + \frac{1}{\delta}c^2$. We prove (4.3) for $2 < p < \infty$ below.

Assuming (4.3) momentarily, we have from (4.2) that for $T \in \mathcal{T}_{i+1}$

$$\eta_{0,i+1,p}(v_{i+1}, T)^p \leq (1 + \delta)\eta_{0,i+1,p}(v_i, T)^p + C_p \mu_p(\delta) \|v_i - v_{i+1}\|_{L_p(\omega_T)}^p. \quad (4.4)$$

Summing over $T \in \mathcal{T}_{i+1}$ yields

$$\eta_{0,i+1,p}(v_{i+1}, \mathcal{T}_{i+1})^p \leq (1 + \delta)\eta_{0,i+1,p}(v_i, \mathcal{T}_{i+1})^p + C_p \mu_p(\delta) \|v_i - v_{i+1}\|_{L_p(\Omega)}^p.$$

Letting $\tilde{T} \in \mathcal{M}_i$ be a marked element, we have $h_{T'} \leq 2^{-\frac{b}{n}} h_{\tilde{T}}$ for $T' \in \mathcal{T}_{i+1, \tilde{T}}$ and thus from (2.7)

$$\eta_{0,i+1,p}(v_i, \mathcal{T}_{i+1, \tilde{T}})^p \leq 2^{-\frac{(p+1)b}{n}} \eta_{0,i,p}(v_i, \tilde{T})^p.$$

Employing the indicator monotonicity property $\eta_{0,i+1,p}(v_i, \mathcal{T}_{i+1, T}) \leq \eta_{0,i,p}(v_i, T)$ for $T \in \mathcal{T}_i \setminus \mathcal{M}_i$ and summing over $T \in \mathcal{T}_{i+1}$ yields

$$\eta_{0,i+1,p}(v_i, \mathcal{T}_{i+1})^p \leq \eta_{0,i,p}(v_i, \mathcal{T}_i)^p - \lambda_p \eta_{0,i,p}(v_i, \mathcal{M}_i)^p. \quad (4.5)$$

Inserting (4.5) into (4.4) completes the proof of (4.1), assuming (4.3).

To prove (4.3), note first that if $p \geq 2$ is an integer, then for some coefficients α_i we have $(b+c)^p = b^p + c^p + \sum_{1 \leq i \leq p-1} \alpha_i b^i c^{p-i}$. Using Young's inequality $de \leq \frac{d^r}{r} + \frac{e^s}{s}$ for $\frac{1}{r} + \frac{1}{s} = 1$ with $d = (\delta/p)^{i/p} b^i$, $e = (p/\delta)^{i/p} \alpha_i c^{p-i}$, $r = \frac{p}{i}$ and $s = \frac{p}{p-i}$ yields

$$\begin{aligned} a^p &\leq (b+c)^p \leq b^p + c^p + \sum_{1 \leq i \leq p-1} \frac{i\delta}{p^2} b^p + \frac{p-i}{p} \left(\frac{p}{\delta}\right)^{\frac{i}{p-i}} \alpha_i^{\frac{p}{p-i}} c^p \\ &\leq (1+\delta)b^p + C_p \mu_p(\delta) c^p, \end{aligned}$$

where $\mu_p(\delta) = 1 + \delta^{-p+1}$. If $p > 2$, let q be the smallest integer for which $q \geq p$. Recalling the elementary inequality $(d^q + e^q)^{1/q} \leq (d^p + e^p)^{1/p}$ whenever $1 \leq p \leq q \leq \infty$ and $d, e \geq 0$, we set $d = (1+\delta)^{1/q} b$ and $e = [C(q)\mu_q(\delta)]^{1/q} c$ and compute

$$a^p = (a^q)^{\frac{p}{q}} \leq [(1+\delta)b^q + C_q \mu_q(\delta) c^q]^{\frac{p}{q}} \leq (1+\delta)^{\frac{p}{q}} b^p + [C(q)\mu_q(\delta)]^{\frac{p}{q}} c^p.$$

Since $(1+\delta) > 1$ and $p/q \leq 1$, we have $(1+\delta)^{p/q} \leq (1+\delta)$, which completes the proof of (4.3) and thus of Lemma 4.1. \square

4.2. Convergence of AFEM using local marking only. In this section we characterize the effects of marking using only local energy-type estimators. We consider two different marking schemes, one which employs a combination of the ϕ -weighted energy and L_2 indicators in (3.1), and another which uses only standard local energy estimators. Because pollution effects are not controlled, it cannot be expected that the sequence of approximations $\{u_i\}$ produced by such AFEM will converge to the actual solution u , even locally on D or D_d . We are able to characterize with some precision the limiting function u_∞ obtained by such a procedure, however.

THEOREM 4.2. *Assume that for some $0 < \theta < 1$ and all $i \geq 0$, the marked set $\mathcal{M}_i \subset \mathcal{T}_{i, D_d}$ in the module `mark` in §2.7 satisfies one of the following:*

$$(i) \quad \eta_{1,i,\phi}(\mathcal{M}_i)^2 + \eta_{0,i,2}(\mathcal{M}_i \cap \mathcal{T}_{i, D'})^2 \geq \theta^2 (\eta_{1,i,\phi}(\mathcal{T}_{i, D_d})^2 + \eta_{0,i,2}(\mathcal{T}_{i, D'})^2), \quad (4.6)$$

$$(ii) \quad \eta_{1,i}(\mathcal{M}_i)^2 \geq \theta^2 \eta_{1,i}(\mathcal{T}_{i, D_d})^2. \quad (4.7)$$

Then there exists $u_\infty \in H_0^1(\Omega)$ such that $u_i \rightarrow u_\infty$ in $H_0^1(\Omega)$, and furthermore $\mathcal{R}^i \rightarrow \phi u_\infty$ in $H^1(D_d)$. Let also $S_\infty = \overline{\bigcup_{i=0}^\infty S_i}$. Then $u_\infty \in S_\infty$ satisfies $A(u - u_\infty, v) = 0$ for all $v \in S_\infty$, or equivalently, u_∞ is the elliptic projection of u onto S_∞ . Furthermore, $-\Delta u_\infty = f$ in D_d .

Proof. Letting $H = H_0^1(\Omega)$ and $\{S_i\}$ be the Hilbert spaces and subspaces in Lemma 2.6, we have $S_0 \subseteq S_1 \subseteq \dots \subseteq S_i \subseteq \dots \subseteq H_0^1(\Omega)$. Thus by Lemma 2.6 $u_i \rightarrow u_\infty$ in $H_0^1(\Omega)$, where u_∞ is the elliptic projection of u onto S_∞ , as asserted.

Let $0 < \delta < 1$ and $\lambda = 1 - 2^{-\frac{b}{n}}$ be as in Lemma 3.7. Lemma 3.7 and Lemma 4.1 then yield

$$\begin{aligned} & \eta_{1,i+1,\phi}(\mathcal{T}_{i+1,D_d})^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1,D'})^2 \\ & \leq (1 + \delta) [\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 + \eta_{0,i,2}(\mathcal{T}_{i,D'})^2 \\ & \quad - \lambda(\eta_{1,i,\phi}(\mathcal{M}_i)^2 + \eta_{0,i,2}(\mathcal{M}_i \cap \mathcal{T}_{i,D'})^2)] \\ & \quad + C(d)(1 + \frac{1}{\delta})(\|\phi(u_i - u_{i+1})\|_{D_d}^2 + \|u_i - u_{i+1}\|_{L_2(\Omega)}^2). \end{aligned} \quad (4.8)$$

Similarly, from [7] we have

$$\begin{aligned} \eta_{1,i+1}(\mathcal{T}_{i+1,D_d})^2 & \leq (1 + \delta)(\eta_{1,i}(\mathcal{T}_{i,D_d})^2 - \lambda\eta_{1,i}(\mathcal{M}_i)^2) \\ & \quad + C(1 + \frac{1}{\delta})\|u_i - u_{i+1}\|_{\Omega}^2. \end{aligned} \quad (4.9)$$

Inserting (4.6) and (4.7) into (4.8) and (4.9) and then choosing δ so that $\xi^2 = (1 + \delta)(1 - \lambda\theta^2) < 1$ yields

$$\begin{aligned} \eta_{1,i+1,\phi}(\mathcal{T}_{i+1,D_d})^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1,D'})^2 & \leq \xi^2(\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 + \eta_{0,i,2}(\mathcal{T}_{i,D'})^2) \\ & \quad + C(d)(\|\phi(u_i - u_{i+1})\|_{D_d}^2 + \|u_i - u_{i+1}\|_{L_2(\Omega)}^2), \end{aligned} \quad (4.10)$$

$$\eta_{1,i+1}(\mathcal{T}_{i+1,D_d})^2 \leq \xi^2\eta_{1,i}(\mathcal{T}_{i,D_d})^2 + C\|u_i - u_{i+1}\|_{\Omega}^2. \quad (4.11)$$

The convergence of u_i to u_∞ in $H_0^1(\Omega)$ implies that $\{u_i\}$ is Cauchy in $H_0^1(\Omega)$ and by a Poincaré inequality also in $L_2(\Omega)$. Thus the terms $C(\|\phi(u_i - u_{i+1})\|_{D_d}^2 + \|u_i - u_{i+1}\|_{L_2(\Omega)}^2)$ and $C\|u_i - u_{i+1}\|_{\Omega}^2$ converge to 0 as $i \rightarrow \infty$. This fact along with the contraction of the estimator terms in (4.10) and (4.11) then yields $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 + \eta_{0,i,2}(\mathcal{T}_{i,D'})^2 \rightarrow 0$ or $\eta_{1,i}(\mathcal{T}_{i,D_d})^2 \rightarrow 0$ as $i \rightarrow \infty$, respectively, depending on whether (4.6) or (4.7) is used. In the first case, the a posteriori estimate (3.1) yields

$$\|\mathcal{R}^i - \phi u_i\|_{D_d} \rightarrow 0 \text{ as } i \rightarrow \infty. \quad (4.12)$$

Noting that $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 + \eta_{0,i,2}(\mathcal{T}_{i,D'})^2 \leq CL : \eta_{1,i}(\mathcal{T}_{i,D_d})$ yields (4.12) when the marking (4.7) is used. This completes the proof of convergence of the AFEM.

We finally complete the characterization of u_∞ . (4.12) along with the convergence of u_i to u_∞ yields $\mathcal{R}^i \rightarrow \mathcal{R}^\infty = \phi u_\infty$ in $H_0^1(D_d)$. Writing the weak form of (2.9) and then employing the H^1 convergence of u_i to u_∞ and of \mathcal{R}^i to $\mathcal{R}^\infty = \phi u_\infty$ then yields

$$\int_{D_d} \nabla(\phi u_\infty) \nabla v \, dx = \int_{D_d} (\phi f - 2\nabla\phi \nabla u_\infty - u_\infty \Delta\phi) v \, dx, \quad v \in H_0^1(D_d).$$

Expanding the term $\nabla(\phi u_\infty)$ on the left and integrating by parts the last term on the right above yields

$$\begin{aligned} & \int_{D_d} (\nabla u_\infty \nabla(\phi v) - v \nabla u_\infty \nabla\phi + u_\infty \nabla\phi \nabla v) \, dx \\ & = \int_{D_d} (f\phi v - 2v \nabla\phi \nabla u_\infty + v \nabla u_\infty \nabla\phi + u_\infty \nabla\phi \nabla v) \, dx, \end{aligned}$$

which with proper cancellation of terms yields

$$\int_{D_d} \nabla u_\infty \nabla(\phi v) \, dx = \int_{D_d} f(\phi v) \, dx, \quad v \in H_0^1(D_d).$$

Since by assumption $\phi > 0$ in (the interior of) D_d , the density of $C_0^1(D_d)$ in $H_0^1(D_d)$ immediately yields

$$\int_{D_d} \nabla u_\infty \nabla v \, dx = \int_{D_d} f v \, dx, \quad v \in H_0^1(\Omega).$$

Thus $u - u_\infty$ is harmonic in D_d , as asserted. \square

Remark 4.3. When only a local marking such as (4.6) or (4.7) is used, elements outside of \mathcal{T}_{i,D_d} will be refined only to maintain mesh conformity. It is thus expected that some elements in the sequence of meshes $\{\mathcal{T}_i\}$ will not be refined after some finite number of steps in the algorithm, even in regions of $\Omega \setminus D_d$ where further refinement is necessary to resolve u . Thus S_∞ will consist of functions that are piecewise polynomial away from D_d independent of the character of u there, or more precisely, $S_\infty \subsetneq H_0^1(\Omega)$ independent of the properties of u on $\Omega \setminus D_d$. Thus $u_\infty \neq u$ globally. Because u_∞ is the *global* elliptic projection of u onto S_∞ and elliptic solution operators propagate local effects globally, we also expect that $u \neq u_\infty$ locally (i.e., on D or D_d). Thus as stated above, $u - u_\infty$ is harmonic in D_d , but in general not 0. The computational experiment presented in §5.2 confirms that $\|\mathcal{R}^i - \phi u_i\|_{D_d} \rightarrow 0$ as $i \rightarrow \infty$ when local marking is used, but $\phi u_i \not\rightarrow \phi u$ (that is, $u_\infty \neq u$ even in D_d).

Remark 4.4. Essentially the same result as above holds for the marking strategy (4.7) when D_d is taken to be *any* mesh subdomain of Ω relative to the initial mesh \mathcal{T}_0 . In order to obtain this result, we use the alternate definition $-\Delta \tilde{\mathcal{R}}^i = f$ in D_d , $\tilde{\mathcal{R}}^i = u_i$ on ∂D_d of the local reconstruction. Then $\|\tilde{\mathcal{R}}^i - u_i\|_{D_d} \leq C \eta_{1,i}(\mathcal{T}_{i,D_d})$, and the above arguments apply largely unchanged.

Remark 4.5. The inclusion of the local L_2 -type estimator $\eta_{0,i,2}(\mathcal{T}_{i,D'})$ in the marking (4.6) enables us to employ the a posteriori estimate (3.1). Note that $\eta_{0,i,2}(\mathcal{T}_{i,D'})$ is computed entirely locally and *does not* serve as an a posteriori bound for $\|u - u_i\|_{L_2}$ over any subdomain of Ω . In particular, it does not control pollution errors or measure solution effects from outside of D_d . Its inclusion in (3.1) is due entirely to the lack of strict Galerkin orthogonality between \mathcal{R}^i and ϕu_i , as characterized in (2.12).

Remark 4.6. The markings (4.6) and (4.7) may be modified so that edge contributions from $\partial D'$ in $\eta_{0,i,2}$ and from ∂D_d in $\eta_{1,i}$ are excluded. The resulting estimators would include solution and data information *only* from D_d and not from any neighboring elements. Theorem 4.2 has the advantage of using only standard indicators.

4.3. Convergence of an AFEM with adaptive pollution control. In this section we employ in an alternating fashion the Dörfler marking (2.14) for the ϕ -weighted local energy error and a similar bulk criterion for the global L_p error, with the marking chosen in a given step depending upon which estimator dominates. We will thus obtain convergence of an AFEM employing a marking strategy that can be used to effectively control pollution errors on arbitrary polyhedral domains Ω . We do not obtain a rate of convergence for this AFEM in the general case. In the following subsection we show that this marking yields a contraction under substantially more restrictive assumptions.

We first describe our marking strategy. Let $\beta > 0$ and $0 < \theta < 1$ be given, and let p be chosen as in Lemma 2.2 so that $\|u - u_i\|_{L_p(\Omega)} \leq C_p \eta_{0,i,p}(\mathcal{T}_i)$. At each step of the algorithm, we determine $\mathcal{M}_i \subset \mathcal{T}_i$ so that:

1. If $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) \geq \beta\eta_{0,i,p}(\mathcal{T}_i)$,

$$\eta_{1,i,\phi}(\mathcal{M}_i) \geq \theta\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}). \quad (4.13)$$

2. Otherwise (that is, if $\beta\eta_{0,i,p}(\mathcal{T}_i) > \eta_{1,i,\phi}(\mathcal{T}_{i,D_d})$),

$$\eta_{0,i,p}(\mathcal{M}_i) \geq \theta\eta_{0,i,p}(\mathcal{T}_i). \quad (4.14)$$

We now prove that the adaptive algorithm defined above is convergent.

THEOREM 4.7. *Assume that $p = 2$ if Ω is convex and $4 < p < 6$ otherwise. Then the adaptive algorithm with marking strategy given by (4.13) and (4.14) yields*

$$\|\phi(u - u_i)\|_{D_d} + \eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + \|u - u_i\|_{L_p(\Omega)} + \eta_{0,i,p}(\mathcal{T}_i) \rightarrow 0 \text{ as } i \rightarrow \infty. \quad (4.15)$$

Proof. Let $C(d)$ be a generic constant which may depend on p and d as well as other nonessential quantities. Assume first that $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) \geq \beta\eta_{0,i,p}(\mathcal{T}_i)$. We then have from (4.13) that $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 - \lambda\eta_{1,i,\phi}(\mathcal{M}_i)^2 \leq (1 - \lambda\theta^2)\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2$. Choosing δ in (3.16) so that $\tilde{\alpha}^2 := (1 + \delta)(1 - \lambda\theta^2) < 1$, noting that $(a^2 + b^2)^{1/2} \leq a + b$, and bounding the terms in (3.16) involving $u_i - u_{i+1}$ by $\|u_i - u_{i+1}\|_{\Omega}^2$, we have from (3.16) that

$$\eta_{1,i+1,\phi}(\mathcal{T}_{i+1,D_d}) \leq \tilde{\alpha}\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + C(d)\|u_i - u_{i+1}\|_{\Omega}.$$

Employing (4.1) while noting the Sobolev inequality $\|u_i - u_{i+1}\|_{L_p(\Omega)} \leq C\|u_i - u_{i+1}\|_{\Omega}$ for $4 < p < 6$ when $n = 2, 3$ and also noting that $(a^p + b^p)^{1/p} \leq a + b$, we thus have for $\delta > 0$

$$\begin{aligned} & \eta_{1,i+1,\phi}(\mathcal{T}_{i+1,D_d}) + \beta\eta_{0,i+1,p}(\mathcal{T}_{i+1}) \\ & \leq \tilde{\alpha}\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + (1 + \delta)\beta\eta_{0,i,p}(\mathcal{T}_i) + C(d)\mu_p(\delta)^{1/p}\|u_i - u_{i+1}\|_{\Omega}. \end{aligned}$$

Choosing $\delta = \frac{1 - \tilde{\alpha}}{3}$ and letting $\alpha = \frac{2 + \tilde{\alpha}}{3} < 1$, we have $1 + \delta = \alpha + 2\delta$ and $\tilde{\alpha} + 2\delta = \alpha$. Noting from (4.13) that $2\delta\beta\eta_{0,i,p}(\mathcal{T}_i) \leq 2\delta\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})$, we thus have

$$\begin{aligned} & \eta_{1,i+1,\phi}(\mathcal{T}_{i+1,D_d}) + \beta\eta_{0,i+1,p}(\mathcal{T}_{i+1}) \\ & \leq \alpha(\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + \beta\eta_{0,i,p}(\mathcal{T}_i)) + C(d)\|u_i - u_{i+1}\|_{\Omega}. \end{aligned} \quad (4.16)$$

(4.16) also holds in the case when $\beta\eta_{0,i,p}(\mathcal{T}_i) > \eta_{1,i,\phi}(\mathcal{T}_{i,D_d})$; we omit the details.

From Lemma 2.6, we see that $\{u_i\}$ is a convergent sequence in $H_0^1(\Omega)$ and is thus Cauchy. Thus $\|u_i - u_{i+1}\|_{\Omega} \rightarrow 0$ as $i \rightarrow \infty$. Using this observation in (4.16) along with the fact that $\alpha < 1$ yields $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + \beta\eta_{0,i,p}(\mathcal{T}_i) \rightarrow 0$ as $i \rightarrow \infty$. Recalling that $\|\phi(u - u_i)\|_{D_d} + \|u - u_i\|_{L_p(\Omega)} \leq C[\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + \beta\eta_{0,i,p}(\mathcal{T}_i)]$ finally yields (4.15). \square

Remark 4.8. The argument above in fact yields $u_i \rightarrow u$ in $H_0^1(\Omega)$ under the marking (4.13)–(4.14). That is, *global* convergence of u_i to u in the energy norm is achieved even though our adaptive method only seeks to control the energy error locally. As in the preceding subsection, the marking (4.13) can also be replaced in Theorem 4.7 by marking with a standard local energy estimator over any subdomain, so that employing the ϕ -weighted error notion is not essential for obtaining plain convergence. These observations highlight the fact that Theorem 4.7 is a plain convergence result yielding no information about the rate of convergence, and is in this sense rather rough.

Remark 4.9. A general framework for obtaining plain convergence results for AFEM is developed in [22] for energy norms and extended to weaker norms such as

those considered here in [21]. The alternating marking strategy that we employ here does not satisfy the marking assumption made in [21] (essentially, that the element with the largest indicator is marked in each refinement step), but it does satisfy the more general criterion regarding the “mark” step given in §5 of [22]. The L_p estimators used here can also be fit into the framework of [21] by slightly altering the jump term. Thus with proper alteration of some technical details, the general framework of [21] and [22] could likely be applied to obtain a plain convergence result for the error notion $|||\phi(\cdot)|||_{D_d} + \|\cdot\|_{L_p(\Omega)}$, but we do not pursue the details. Employing the framework of [21] would allow for more general marking strategies such as an alternating maximum strategy instead of the alternating Dörfler marking employed here.

4.4. A contractive AFEM for controlling $|||\phi(u - u_i)|||_{D_d}$. We finally establish a contraction property for an AFEM based on the marking (4.13)–(4.14), but only under significantly more restrictive assumptions. First, we assume that Ω is convex so that we may choose $p = 2$ in our estimates. Secondly, the parameter β in the marking strategy (4.13)–(4.14) must be sufficiently large. Finally, our result requires a restriction on the family of meshes $\{\mathcal{T}_i\}$ beyond shape regularity.

We first discuss properties of shape regular meshes in the context of finite element error estimation. The discussion here is an abbreviated version of that contained in the work [11] in which optimality of an AFEM for controlling $\|u - u_i\|_{L_2(\Omega)}$ is established. We refer the reader to that work for more details.

When measuring errors in the global energy norm, standard a priori theory yields

$$\|u - u_i\|_{\Omega} \leq \min_{\chi \in S_i} \|u - \chi\|_{\Omega} \leq C \|h^k D^{k+1} u\|_{L_2(\Omega)}, \quad (4.17)$$

where h is the local mesh size. Note that (4.17) optimally reflects mesh grading. Letting $\bar{h} = \max_{T \in \mathcal{T}_i} h_T$, a standard duality argument yields in contrast

$$\|u - u_i\|_{L_2(\Omega)} \leq C \bar{h} \min_{\chi \in S_i} \|u - \chi\|_{\Omega} \leq C \bar{h} \|h^k D^{k+1} u\|_{L_2(\Omega)}. \quad (4.18)$$

(4.18) *does not optimally reflect mesh grading* because of the factor of \bar{h} lying outside of the norm. A longstanding open problem in a priori error analysis is to prove estimates in the L_2 and L_∞ norms which assume only shape-regular meshes and which optimally reflect mesh grading as in (4.17). Such estimates hold in one space dimension (cf. [1]), but have not been proved for $n \geq 2$.

It can be shown (cf. [24]) that for any shape-regular mesh \mathcal{T}_i , there exists a regularized mesh function $h_i \in W_\infty^1(\Omega)$ which is equivalent to the local mesh size h_T up to a constant, and which satisfies $\|\nabla h_i\|_{L_\infty(\Omega)} \leq C$, where C depends only on the shape regularity of \mathcal{T}_i . We will assume that the sequence of meshes $\{\mathcal{T}_i\}$ is *mildly graded* in the sense that for each \mathcal{T}_i , $i \geq 0$, we can construct a continuous, piecewise linear mesh function h_i such that

$$c_{\mathcal{T}} h_i(x) \leq h_T \leq C_{\mathcal{T}} h_i(x), \quad x \in T \in \mathcal{T}_i, \quad (4.19)$$

$$\|\nabla h_i\|_{L_\infty(\Omega)} \leq \mu, \quad \mu \text{ sufficiently small}, \quad (4.20)$$

$$h_{i+1}(x) \leq h_i(x), \quad x \in \Omega. \quad (4.21)$$

Here μ , $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$ do not depend on i . [13] contains quasi-optimal a priori L_∞ estimates proved assuming a condition similar to (4.19)–(4.20). Optimal L_2 estimates in one space dimension are proved in the introductory chapter of the standard reference [6] assuming (4.19)–(4.20); extension to higher space dimensions is not difficult. Note

also that (4.21) can be enforced artificially and does not add a substantial restriction to the class of allowed meshes.

Simple counterexamples show that not all shape regular meshes are mildly graded, but it is shown in [13] that it is possible to construct mildly graded meshes which for example optimally resolve corner singularities. Adaptive methods do not necessarily produce mildly graded meshes, but in [11] we show that given μ , the standard adaptive bisection algorithm can be modified so that (4.19)–(4.21) are enforced in a way that does not compromise optimality properties of adaptive methods.

The conditions (4.19)–(4.20) allow us to in essence use the error notion $|||h_i(u - u_i)|||_\Omega$ as a proxy for the L_2 error $\|u - u_i\|_{L_2(\Omega)}$. The following results related to this error notion are proved in [11].

LEMMA 4.10. *Assume that Ω is a convex polyhedral domain, and that (4.19)–(4.21) hold. Then the following a posteriori estimate holds:*

$$\|u - u_i\|_{L_2(\Omega)} + |||h_i(u - u_i)|||_\Omega \leq C\eta_{0,i,2}(\mathcal{T}_i). \quad (4.22)$$

In addition, there holds for any $\Lambda > 1$ the following quasi-orthogonality estimate:

$$\begin{aligned} |||h_{i+1}(u - u_{i+1})|||_\Omega^2 &\leq \Lambda |||h_i(u - u_i)|||_\Omega^2 - |||h_{i+1}(u_{i+1} - u_i)|||_\Omega^2 \\ &\quad + \frac{C_5\mu^2}{\Lambda - 1} (\eta_{0,i,2}(\mathcal{T}_i)^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2). \end{aligned} \quad (4.23)$$

Finally, the following estimator reduction property holds. Let \mathcal{T}_{i+1} be obtained by bisecting $\mathcal{M}_i \subset \mathcal{T}_i$ at least b times. Then for any $0 < \delta \leq 1$,

$$\begin{aligned} \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2 &\leq (1 + \delta)(\eta_{0,i,2}(\mathcal{T}_i)^2 - \lambda_2\eta_{0,i,2}(\mathcal{M}_i)^2) \\ &\quad + C_6\left(1 + \frac{1}{\delta}\right) \left[|||h_{i+1}(v_i - v_{i+1})|||_\Omega^2 + \mu^2(\eta_{0,i,2}(\mathcal{T}_i)^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2)\right]. \end{aligned}$$

We finally prove the following contraction result.

THEOREM 4.11. *Given $\gamma > 0$, define $E_{1,i,\phi}^2 = |||\phi(u - u_i)|||_{D_d}^2 + \gamma\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2$ and $E_{0,i,2}^2 = |||h_i(u - u_i)|||_\Omega^2 + \gamma\eta_{0,i,2}(\mathcal{T}_i)^2$. Assume also that Ω is convex and polyhedral and that the parameter β in the marking criterion (4.13) and (4.14) satisfies $\beta \geq \frac{\kappa}{d}$ with κ sufficiently large. Assume in addition that there exist mesh functions h_i and h_{i+1} satisfying (4.19), (4.20), and (4.21) with μ in (4.20) being sufficiently small. Then there exist constants $\gamma > 0$ and $0 < \alpha < 1$ such that*

$$E_{1,i+1,\phi}^2 + \beta^2 E_{0,i+1,2}^2 \leq \alpha^2 (E_{1,i,\phi}^2 + \beta^2 E_{0,i,2}^2). \quad (4.24)$$

Here the constants α and γ , the maximum allowable size of μ , and the minimum allowable size of κ depend upon λ , θ , and the generic constant C in §2.5.

Proof. Let $E_i = |||\phi(u - u_i)|||_{D_d}^2 + \beta^2 |||h_i(u - u_i)|||_\Omega^2$, $\eta_i = \eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 + \beta^2\eta_{0,i,2}(\mathcal{T}_i)^2$, and $e_i = |||\phi(u_i - u_{i+1})|||_{D_d}^2 + \beta^2 |||h_{i+1}(u_i - u_{i+1})|||_\Omega^2$. Note that $\lambda \leq \lambda_2$, where λ_2 and λ are defined in Lemma 4.1 and Lemma 3.7, respectively. The dual marking strategy (4.13)–(4.14) thus yields

$$\eta_i^2 - \lambda(\eta_{1,i,\phi}(\mathcal{M}_i)^2 + \beta^2\eta_{0,i,2}(\mathcal{M}_i)^2) \leq \left(1 - \frac{\lambda\theta^2}{2}\right) \eta_i^2. \quad (4.25)$$

Employing in turn the quasi-orthogonality properties (3.11) and (4.23) along with the a posteriori estimate (4.22) while adjusting constants, the estimator reduction

properties (4.1) and (3.16), and (4.25) yields for any $\Lambda > 1$, $\tilde{\gamma} > 0$, and $0 < \delta \leq 1$

$$\begin{aligned} E_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 &\leq \Lambda E_i^2 - e_i^2 + \tilde{\gamma}\eta_{i+1}^2 \\ &\quad + \frac{C_7}{\Lambda - 1}(d^{-2} + \mu^2\beta^2)(\eta_{0,i,2}(\mathcal{T}_i)^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2) \\ &\leq \Lambda E_i^2 + \tilde{\gamma}(1 + \delta)\left(1 - \frac{\lambda\theta^2}{2}\right)\eta_i^2 + \left(\tilde{\gamma}C_8\left(1 + \frac{1}{\delta}\right) - 1\right)e_i^2 \\ &\quad + \left(\frac{C_7}{\Lambda - 1} + \tilde{\gamma}C_8\left(1 + \frac{1}{\delta}\right)\right)(d^{-2} + \mu^2\beta^2)(\eta_{0,i,2}(\mathcal{T}_i)^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2). \end{aligned}$$

We next choose $\tilde{\gamma}$ so that

$$\tilde{\gamma} = \frac{\delta}{C_8(1 + \delta)}, \quad \tilde{\gamma}C_8\left(1 + \frac{1}{\delta}\right) = 1,$$

and in addition specify that

$$\mu \leq \frac{1}{d\beta}. \quad (4.26)$$

Then

$$\begin{aligned} E_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 &\leq \Lambda E_i^2 + \tilde{\gamma}(1 + \delta)\left(1 - \frac{\lambda\theta^2}{4}\right)\eta_i^2 - \tilde{\gamma}(1 + \delta)\frac{\lambda\theta^2}{4}\eta_i^2 \\ &\quad + 2d^{-2}\left(\frac{C_7}{\Lambda - 1} + 1\right)(\eta_{0,i,2}(\mathcal{T}_i)^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2). \end{aligned} \quad (4.27)$$

We now require that $\beta \geq \frac{1}{d}$ (i.e., $\kappa \geq 1$), so that from (3.4) and (4.22) we have $E_i^2 \leq C_9\eta_i^2$. Inserting this relationship into (4.27) while noting that $\tilde{\gamma}(1 + \delta) = \frac{\delta}{C_8}$ yields

$$\begin{aligned} E_{i+1}^2 + \tilde{\gamma}\eta_{i+1}^2 &\leq \left(\Lambda - \frac{\lambda\theta^2\delta}{4C_8C_9}\right)E_i^2 + \tilde{\gamma}(1 + \delta)\left(1 - \frac{\lambda\theta^2}{4}\right)\eta_i^2 + \\ &\quad + 2d^{-2}\left(\frac{C_7}{\Lambda - 1} + 1\right)(\eta_{0,i,2}(\mathcal{T}_i)^2 + \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2). \end{aligned} \quad (4.28)$$

Next we choose $\Lambda > 1$ and $\delta > 0$ so that

$$\tilde{\alpha}^2 = \max\left\{\Lambda - \frac{\lambda\theta^2\delta}{4C_8C_9}, (1 + \delta)\left(1 - \frac{\lambda\theta^2}{4}\right)\right\} < 1. \quad (4.29)$$

Our choice of Λ and δ satisfying (4.29) may depend on the generic constant C , θ , and λ , but is independent of β and d . Similarly, $\tilde{\gamma}$ depends only on C , θ , and λ . Inserting (4.29) into (4.28), defining $C_{10} = \frac{2C_7}{\Lambda - 1} + 2$, and rearranging yields

$$\begin{aligned} E_{i+1}^2 + \tilde{\gamma} &\left(\eta_{1,i+1,\phi}(\mathcal{T}_{i+1,D_d})^2 + \left(\beta^2 - \frac{C_{10}}{\tilde{\gamma}d^2}\right)\eta_{0,i+1,2}(\mathcal{T}_{i+1})^2\right) \\ &\leq \tilde{\alpha}^2 \left[E_i^2 + \tilde{\gamma} \left(\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})^2 + \left(\beta^2 + \frac{C_{10}}{\tilde{\gamma}\tilde{\alpha}^2d^2} \right) \eta_{0,i,2}(\mathcal{T}_i)^2 \right) \right]. \end{aligned}$$

Let $\gamma = \tilde{\gamma} - \frac{C_{10}}{\beta^2d^2} = \tilde{\gamma} - \frac{C_{10}}{\kappa^2}$ and $\alpha^2 = \frac{\tilde{\gamma}}{\gamma}\tilde{\alpha}^2 + \frac{C_{10}}{\gamma\beta^2d^2} = \frac{\tilde{\gamma}}{\gamma}\tilde{\alpha}^2 + \frac{C_{10}}{\gamma\kappa^2}$, so that $\gamma\alpha^2 = \tilde{\gamma}\tilde{\alpha}^2 + \frac{C_{10}}{\kappa^2}$. We choose κ sufficiently large to ensure that $\gamma > 0$ and $0 < \alpha^2 < 1$. Note that our

choice of κ and thus of α and γ depends only on C , θ , and λ , but not on d . Noting that $\gamma < \tilde{\gamma}$, $\tilde{\alpha}^2 < \alpha^2$, and $\tilde{\gamma}\tilde{\alpha}^2 < \gamma\alpha^2$, we have

$$\begin{aligned} E_{i+1}^2 + \gamma\eta_{i+1}^2 &\leq E_{i+1}^2 + \tilde{\gamma} \left(\eta_{1,i+1,\phi}(\mathcal{T}_{i+1})^2 + \left(\beta^2 - \frac{C_{10}}{\tilde{\gamma}d^2} \right) \eta_{0,i+1,2}(\mathcal{T}_{i+1})^2 \right) \\ &\leq \tilde{\alpha}^2 \left[E_i^2 + \tilde{\gamma} \left(\eta_{1,i,\phi}(\mathcal{T}_i)^2 + \left(\beta^2 + \frac{C_{10}}{\tilde{\gamma}\tilde{\alpha}^2 d^2} \right) \eta_{0,i,2}(\mathcal{T}_i)^2 \right) \right] \\ &= \tilde{\alpha}^2 E_i^2 + \tilde{\gamma}\tilde{\alpha}^2 \eta_{1,i,\phi}(\mathcal{T}_i)^2 + \beta^2 \left(\tilde{\gamma}\tilde{\alpha}^2 + \frac{C_{10}}{\kappa^2} \right) \eta_{0,i,2}(\mathcal{T}_i)^2 \\ &\leq \alpha^2 (E_i^2 + \gamma\eta_i^2). \end{aligned}$$

Finally recall that we required in (4.26) that $\mu \leq \frac{1}{d\beta} = \frac{1}{\kappa}$. Thus μ also depends on C , λ , and θ , but not on d . This completes the proof of Theorem 4.11. \square

Remark 4.12. The condition that β must be sufficiently large in order to obtain the contraction (4.24) accurately reflects the coupling between $|||\phi(u - u_i)|||_{D_d}$ and $\|u - u_i\|_{L_2(\Omega)}$. As we demonstrate computationally in the following section, the local energy error ceases to decrease beyond a certain number of adaptive iterations if the pollution error is not controlled. If β is too small, then only the local energy marking will be used for a large number of initial adaptive iterations. During these iterations, $\eta_{1,i,\phi}$ will be reduced, but beyond some point the local energy error $|||\phi(u - u_i)|||_{D_d}$ will cease to decrease. Thus a stepwise contraction can not be expected. Plain convergence (without a rate) can still be expected as in Theorem 4.7, however.

5. Computational Experiments. In this section we briefly describe some computational experiments illustrating Theorem 3.8 and Theorem 4.7.

5.1. Computational parameters. Let $\Omega = (-1, 1) \times (-1, 1) \setminus [0, 1] \times \{0\}$. With t and θ denoting polar coordinates about the origin, let $\xi(t, \theta) = t^{1/2} \sin \frac{\theta}{2}$ and $u(x, y) = \cos \frac{\pi x}{2} \cos \frac{\pi y}{2} \xi(t(x, y), \theta(x, y))$. Note that $u|_{\partial\Omega} = 0$ and that ξ is the (most) singular function which naturally arises in expansions of solutions of (1.1) into regular and singular parts. Let also $f = -\Delta u$.

Let $D = (\frac{1}{4}, \frac{3}{4}) \times (0, \frac{1}{4})$, $D_d = (0, 1) \times (0, \frac{1}{2})$, and $D' = D_d \setminus D$. With

$$\phi_x(x) = \begin{cases} 1 - 16(x - \frac{1}{4})^2, & 0 < x < \frac{1}{4}, \\ 1, & \frac{1}{4} \leq x \leq \frac{3}{4}, \\ 1 - 16(x - \frac{3}{4})^2, & \frac{3}{4} < x < 1, \\ 0 & \text{otherwise;} \end{cases} \quad \phi_y(y) = \begin{cases} 1, & 0 < y \leq \frac{1}{4}, \\ 1 - 16(y - \frac{1}{4})^2, & \frac{1}{4} < y < \frac{1}{2}, \\ 0 & \text{otherwise,} \end{cases}$$

we let $\phi(x, y) = \phi_x(x)\phi_y(y)$ and note that D , D_d , and ϕ satisfy the assumptions of §2.2. Finite element spaces of degree $k = 1$ (piecewise linear elements) were employed, and all examples were coded using the finite element toolbox ALBERTA ([25]). Finally, the pollution error was measured in the L_5 norm, i.e., $p = 5$ in (2.13). Ω , D , and D' are depicted in Figure 5.1, along with the initial mesh \mathcal{T}_0 used for all computations.

5.2. Experiment 1: Reduction of $|||\mathcal{R}^i - \phi u_i|||_{D_d}$. In this experiment we illustrate the effectiveness of the local reconstruction \mathcal{R}^i in splitting the overall local error $\phi(u - u_i)$ into a local energy term $\mathcal{R}^i - \phi u_i$ and a pollution term $\mathcal{R}^i - \phi u$. Here we mark *only* using the Dörfler marking (4.13) based on $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})$, with $\theta = 0.5$. Reduction of the pollution error $\|u - u_i\|_{L_p(\Omega)}$ and of the overall local energy error $|||\phi(u - u_i)|||_{D_d}$ can thus not be expected. As proved in Theorem 4.2, however, an appropriate local marking does yield $|||\mathcal{R}^i - \phi u_i|||_{D_d} \rightarrow 0$ as $i \rightarrow \infty$ even if

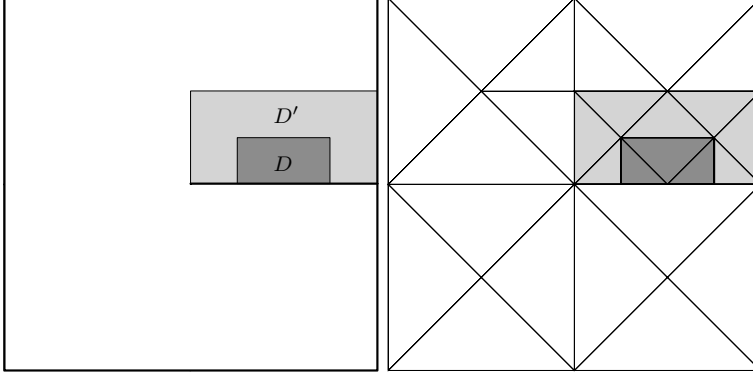


FIG. 5.1. Left: Ω (left) along with D (dark shading), D' (light shading), and $D_d = D \cup D'$. Right: The initial mesh \mathcal{T}_0 .

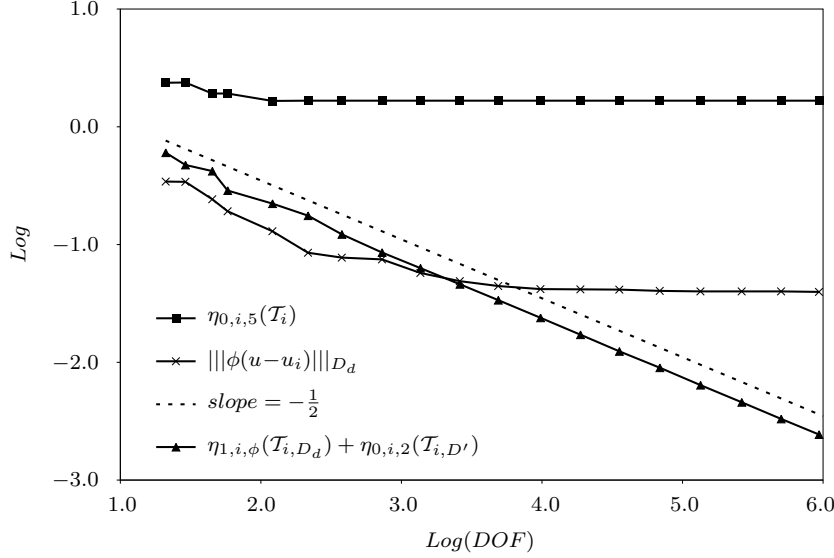


FIG. 5.2. Optimal-order reduction of the reconstruction error $\|\mathcal{R}^i - \phi u_i\|_{D_d} \leq C(\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + \eta_{0,i,2}(\mathcal{T}_{i,D'}))$ using a local energy marking strategy only.

$\|u - u_i\|_{L_p(\Omega)}$ does not vanish as $i \rightarrow \infty$. Note also that while (4.6) of Theorem 4.2 requires a marking based on $\eta_{0,i,2}(\mathcal{T}_{i,D'})$ as well as on $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d})$, the additional marking based on $\eta_{0,i,2}(\mathcal{T}_{i,D'})$ is not observed to be necessary in our example.

In Figure 5.2, we see clearly that indeed $\eta_{1,i,\phi}(\mathcal{T}_{i,D_d}) + \eta_{0,i,2}(\mathcal{T}_{i,D'})$ (which by Theorem 3.1 bounds $\|\mathcal{R}^i - \phi u_i\|_{D_d}$ up to a constant) converges with optimal order using only a local marking strategy. On the other hand, the pollution estimator $\eta_{0,i,5}(\mathcal{T}_i)$ and the overall local error $\|\phi(u - u_i)\|_{D_d}$ do not decrease to 0 as $i \rightarrow \infty$. One of the adaptive meshes produced in this experiment is displayed in Figure 5.3. Note that some elements in $\Omega \setminus D_d$ are refined in this example, but only in order to maintain mesh conformity.

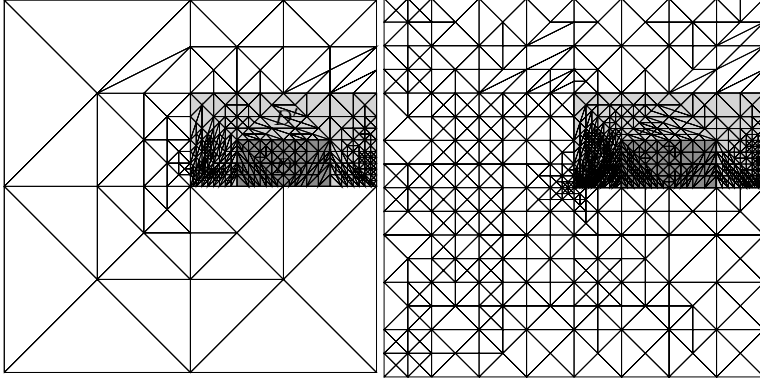


FIG. 5.3. *Left: Adaptive mesh with 376 degrees of freedom produced in Experiment 1 by marking only elements in D_d for refinement; Right: Adaptive mesh with 804 degrees of freedom produced in Experiment 2 by marking for the local energy and global pollution errors.*

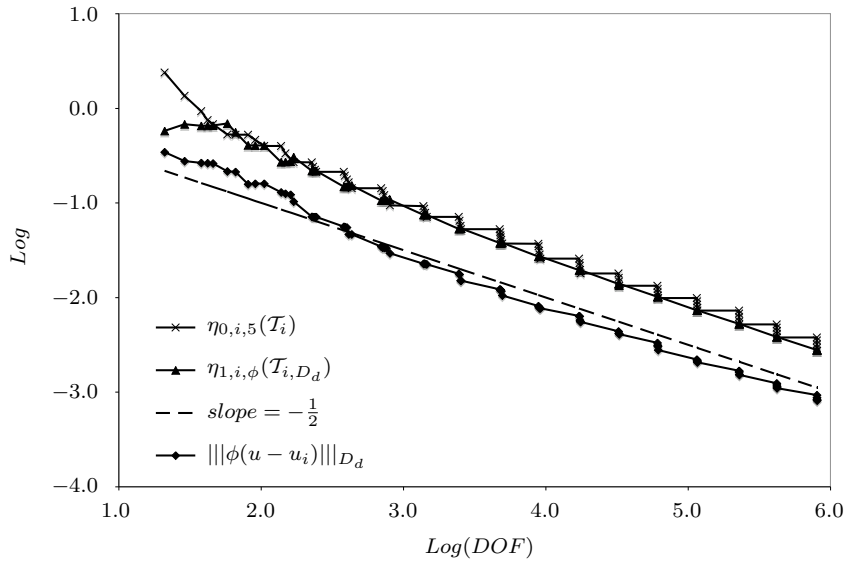


FIG. 5.4. *Optimal-order reduction of the local error $||\phi(u - u_i)||_{D_d}$.*

5.3. Experiment 2: Reduction of $||\phi(u - u_i)||_{D_d}$. In this experiment we apply the full adaptive algorithm of §2.7 to the test problem described above. We let $\theta = 0.7$ and $\beta = 1$ in (4.13) and (4.14). Optimal-order reduction of $||\phi(u - u_i)||_{D_d}$ is observed in Figure 5.4, and an adaptive mesh is displayed in Figure 5.3. A relatively large number of adaptive iterations were required here to reduce the pollution error. A more efficient algorithm can be obtained either by using the maximum strategy when marking for the pollution error, or by choosing the parameters θ in (4.13) and (4.14) differently (e.g., letting $\theta = 0.5$ in (4.13) but $\theta = 0.8$ in (4.14)).

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