HIGHER-ORDER FINITE ELEMENT METHODS AND POINTWISE ERROR ESTIMATES FOR ELLIPTIC PROBLEMS ON SURFACES *

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Abstract. We define higher-order analogs to the piecewise linear surface finite element method studied in [Dz88] and prove error estimates in both pointwise and L_2-based norms. Using the Laplace-Beltrami problem on an implicitly defined surface Γ as a model PDE, we define Lagrange finite element methods of arbitrary degree on polynomial approximations to Γ which likewise are of arbitrary degree. Then we prove a priori error estimates in the L_2, H^1, and corresponding pointwise norms that demonstrate the interaction between the "PDE error" that arises from employing a finite-dimensional finite element space and the "geometric error" that results from approximating Γ. We also consider parametric finite element approximations that are defined on Γ and thus induce no geometric error. Computational examples confirm the sharpness of our error estimates.

Key words. Laplace-Beltrami operator, surface finite element methods, a priori error estimates, boundary value problems on surfaces, pointwise and maximum norm error estimates

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1. Introduction.

The numerical solution of partial differential equations defined on surfaces arises naturally in many applications (cf. [CDR03], [CDDRR04], [BMN05], [He06], and [DE07a], among many others). We consider the following model problem in order to focus on basic issues arising in the definition and analysis of such numerical methods. Let Γ be a smooth n-dimensional surface ($n = 2, 3$) without boundary embedded in $\mathbb{R}^{n+1}$. Let f be given data satisfying $\int_\Gamma f \, d\sigma = 0$ where $d\sigma$ is surface measure, and let u solve

$$-\Delta_\Gamma u = f \text{ on } \Gamma.$$

Here Δ_Γ is the Laplace-Beltrami operator on Γ, and we require $\int_\Gamma u \, d\sigma = 0$ in order to guarantee uniqueness.

Several methods for defining suitable triangulations of Γ and corresponding finite element spaces have been proposed. For example, one may use the manifold structure of Γ (cf. [Ho01]) or a global parametric representation (cf. [AP05]) to triangulate Γ. In this work we focus on the method originally considered in [Dz88] in which Γ is represented as a level set of a smooth signed distance function d. In [Dz88], Γ is approximated by a polyhedral surface Γ_h having triangular faces, and the equations for defining a piecewise linear finite element approximation to u are conveniently defined and solved on Γ_h. This method has several advantages when compared with approaches relying on global or local parametrizations of Γ. These include its flexibility in handling various surfaces and its direct extension to problems in which the surface under consideration evolves in an unknown fashion and a parametrization is thus not available. The paradigm example of such an evolution problem is motion of a surface by mean curvature flow; cf. [Dz91], [DDE05].

In the present work we focus on two goals. The first is to define higher-order analogs to the surface finite element method defined in [Dz88]. Higher-order approximations are desirable in many situations because of their increased computational

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efficiency versus piecewise linear finite element methods. In order to obtain such approximations, it is generally necessary to approximate Γ to higher order in addition to employing higher-order finite element spaces. We thus construct parametric finite element spaces of arbitrary degree that are defined on arbitrary-degree polynomial approximations to Γ. In addition, we describe fully parametric finite element spaces defined directly on Γ via local transformations from the faces of Γ_h so that no error arises from approximating Γ. It should be noted that in both of these cases, we require explicit knowledge of the distance function d (either through an analytical formula or by a numerical approximation) in order to construct our algorithm.

Our second main goal is to carry out a thorough error analysis for finite element methods for the Laplace-Beltrami operator on surfaces. The original work of Dziuk in [Dz88] contains proofs of optimal-order convergence of the piecewise linear surface finite element method in the L_2 and energy norms. Here we prove optimal-order estimates for pointwise errors in function values and gradients and for local energy errors in addition to the L_2 and energy errors. These estimates are valid for arbitrary degrees of finite element spaces and polynomial approximations to Γ. As in [Dz88], we split the overall error into a "geometric error" arising from the approximation of Γ and a standard finite element "almost-best-approximation" error which arises from approximating an infinite-dimensional function space by a finite-dimensional finite element space. Roughly speaking, when employing finite element spaces of degree r on polynomial surface approximations of degree k, we have

$$\begin{aligned}\|\nabla_\Gamma(u-u_h)\|_{L_2(\Gamma)} \leq & Ch^r\|u\|_{H^{r+1}(\Gamma)} + Ch^{k+1}\|u\|_{H^1(\Gamma)},\\ \|u-u_h\|_{L_2(\Gamma)} \leq & Ch^{r+1}\|u\|_{H^{r+1}(\Gamma)} + Ch^{k+1}\|u\|_{H^1(\Gamma)},\end{aligned}$$

where u_h is the finite element solution, ∇_Γ is the tangential gradient on Γ, and C depends on geometric properties of Γ. We also prove similar estimates in L_∞ and W^1_∞. As we verify via numerical experiments, one must thus choose $k+1 \geq r$ to achieve optimal-order convergence in W^1_p norms and $k \geq r$ to achieve optimal-order convergence in L_p norms.

We finally note that approximating Γ via higher-degree polynomials has the added benefit that the curvatures of the approximating surface Γ_h have a natural pointwise definition and converge to those of Γ. The availability of a simple curvature approximation is beneficial in applications where the weak form of the PDE under consideration, and thus also the finite element method, explicitly employs curvature information (as for example in the image processing application in [CDR03]). Curvature information also was used in the a posteriori error estimates given in [DD07]. However, pointwise curvatures are not naturally defined on the piecewise linear discrete surfaces employed in [Dz88], and ad-hoc reconstruction methods must be used to define suitable curvatures if they are explicitly required in calculations (cf. [CDR03]).

An outline of the paper is as follows. §2 contains definitions and preliminaries. In §3 we prove abstract error estimates in various norms. In §4, we demonstrate how these abstract estimates may be applied to various finite element methods on surfaces and give computational results illustrating the basic error behavior of the methods. In §5 we give a brief discussion of conditions under which our error analysis may be extended to more general classes of PDE on surfaces and manifolds.

2. Preliminaries. In this section we record a number of preliminaries concerning geometry, transformations of functions between the continuous and discrete surfaces Γ and Γ_h, analytical results, and finite element approximation theory.

2.1. Geometric and analytical preliminaries on Γ. We assume throughout that Γ is a compact, oriented, C_∞, two- or three-dimensional surface without boundary which is embedded in $\mathbb{R}^3$ or $\mathbb{R}^4$, respectively. Our results may be extended to higher-dimensional surfaces of codimension one if appropriate results from finite element approximation theory can be proved; we restrict ourselves to lower-dimensional manifolds so that we may employ the Lagrange interpolant in our analysis.

Let d be the oriented distance function for Γ. For concreteness, let $d < 0$ on the interior of Γ and $d > 0$ on the exterior of Γ. $\vec{\nu} = \nabla d$ is then the outward-pointing unit normal, and $\mathbf{H} = \nabla^2 d$ is the Weingarten map. Here we express these quantities in the coordinates of the embedding space $\mathbb{R}^{n+1}$ $(n = 2, 3)$. For $x \in \Gamma$, the n eigenvalues $\kappa_1, ..., \kappa_n$ of $\mathbf{H}$ corresponding to eigenvectors perpendicular to $\vec{\nu}$ are the principal curvatures at x. Let $U \subset \mathbb{R}^{n+1}$ be a strip of width δ about Γ, where $\delta > 0$ is sufficiently small to ensure that the decomposition

$$a(x) = x - d(x)\vec{\nu}(x)$$

onto Γ is unique. We also require that $\delta < \min_{i=1,\dots,n} \frac{1}{\|\kappa_i\|_{L_\infty(\Gamma)}}$; cf. [GT98], §14.6. and [DD07].

Let $\mathbf{P} = \mathbf{I} - \vec{\nu} \otimes \vec{\nu}$ be the projection onto the tangent plane at x, where $\otimes$ is the outer product defined by $(\vec{a} \otimes \vec{b})\vec{c} = \vec{a}\vec{b} \cdot \vec{c}$. Then $\nabla_\Gamma = \mathbf{P}\nabla$ is the tangential gradient, $\mathrm{div}_\Gamma = \nabla_\Gamma \cdot$ is the tangential divergence, and $\Delta_\Gamma = \mathrm{div}_\Gamma \nabla_\Gamma$ is the Laplace-Betrami operator. We shall use standard notation ($H^1(\Gamma)$, $W_p^j(\Gamma)$, etc.) for Sobolev spaces and norms of functions possessing j tangential derivatives lying in L_p.

Next we state some analytical results. Let

$$L(u, v) = \int_\Gamma \nabla_\Gamma u \nabla_\Gamma v \, \mathrm{d}\sigma, \tag{2.1}$$

and let $(\cdot, \cdot)$ be the L_2 inner product over Γ.

LEMMA 2.1. *Let $f \in L_2(\Gamma)$ satisfy $\int_\Gamma f \, \mathrm{d}\sigma = 0$. Then the problem $L(u, v) = (f, v) \ \forall \ v \in H^1(\Gamma)$ has a unique weak solution u satisfying $\int_\Gamma u \, \mathrm{d}\sigma = 0$, and*

$$\|u\|_{H_2^2(\Gamma)} \le C\|f\|_{L_2(\Gamma)}. \tag{2.2}$$

Proof: See [Aub82], Chapter 4 for a proof of existence and uniqueness. (2.2) may be proved by local transformations to subsets of $\mathbb{R}^n$ and a covering argument. □

The proofs of our pointwise error estimates also rely on properties of the Green's function. We denote by $\alpha(x, y)$ the surface distance between $x, y \in \Gamma$.

LEMMA 2.2. *There exists a function $G(x, y)$, unique up to a constant, such that for all functions $\phi \in C^2(\Gamma)$,*

$$\phi(x) = \frac{1}{|\Gamma|} \int_\Gamma \phi \, \mathrm{d}\sigma + \int_\Gamma G(x, y)(-\Delta_\Gamma \phi(y)) \, \mathrm{d}\sigma.$$

In addition, for $x, y \in \Gamma$ with $x \ne y$,

$$G(x, y) \le \begin{cases} C(1 + \log \alpha(x, y)), & n = 2, \\ C\alpha(x, y)^{2-n}, & n > 2. \end{cases} \tag{2.3}$$

Also, let $|\gamma + \beta| > 0$, where γ and β are multiindices. Then

$$|D^\gamma_{\Gamma,y} D^\beta_{\Gamma,x} G(x, y)| \le C\alpha(x, y)^{2-n-|\gamma+\beta|}. \tag{2.4}$$

Proof: Existence of the Green's function G, (2.3), and (2.4) for $1 \le |\alpha| \le 2$ and $|\beta| = 0$ are contained in Theorem 4.13 of [Aub82]. (2.4) may be easily extended to arbitrary α, β with $|\alpha + \beta| > 0$ by using the representation (17) on p. 109 of [Aub82]. □

Finally, let $\gamma_\Gamma > 0$ be the largest positive number such that all balls $B_{\gamma_\Gamma}(x_0) = \{x \in \Gamma : \alpha(x, x_0) < \gamma_\Gamma\}$ of radius γ_Γ map smoothly to domains in $\mathbb{R}^n$. Such a number γ_Γ exists since Γ is a smooth, compact surface.

2.2. The discrete surface Γ_h. Let $\Gamma_h \subset U$ be a polyhedron having triangular faces ($n = 2$) or a polytope having tetrahedral cells ($n = 3$) whose vertices lie on Γ and whose faces (cells) are shape-regular and quasi-uniform of diameter h. We shall denote by $\tilde{\mathcal{T}}_h$ the set of triangular faces of Γ_h and by $\mathcal{T}_h$ the image under a of $\tilde{\mathcal{T}}_h$ (i.e., $\mathcal{T}_h$ consists of curved simplices lying on Γ). Let $\vec{\nu}_h$ be the outward unit normal on Γ_h.

We will analyze finite element methods defined on Γ_h, on Γ, and on higher-order polynomial approximations of Γ, but Γ_h will play a central role in defining and analyzing all of them. From a programming standpoint in particular, Γ_h is fundamental to our methods in that the faces $\tilde{\mathcal{T}}_h$ of Γ_h always constitute the "base" triangulation of Γ, with parametric finite element spaces then being defined over $\tilde{\mathcal{T}}_h$.

2.3. Higher-order polynomial approximations to Γ. Next we describe a family Γ_h^k ($k \ge 1$) of polynomial approximations to Γ. The higher-order finite element spaces we use here are largely described in [He05] and also are similar to the surface element spaces described in [Ne76]. First let $\Gamma_h = \Gamma_h^1$ be a polyhedral approximation to Γ as in the preceding subsection. For $k \ge 2$ and for a given element $\tilde{T} \in \tilde{\mathcal{T}}_h$, let $\phi_1^k, .., \phi_{n_k}^k$ be the Lagrange basis functions of degree k on $\tilde{T}$ corresponding to the nodal points $x^1, ..., x^{n_k}$. For $x \in \tilde{T}$, we then define the discrete projection

$$a_k(x) = \sum_{j=1}^{n_k} a(x^j)\phi_j^k(x).$$

Employing the above definition on each element $\tilde{T} \in \tilde{\mathcal{T}}_h$ yields a continuous piecewise polynomial map on Γ_h. We then define the corresponding discrete surface

$$\Gamma_h^k = \{a_k(x) : x \in \Gamma_h\}.$$

Thus each component of a_k is the Lagrange interpolant of the corresponding component of the projection a restricted to Γ_h. Let $\hat{\mathcal{T}}_h^k$ be the image under a_k of $\tilde{\mathcal{T}}_h$, i.e., for $\hat{T} \in \hat{\mathcal{T}}_h^k$, $\hat{T} = a_k(\tilde{T})$ for some $\tilde{T} \in \tilde{\mathcal{T}}_h$. Let also $\mathcal{T}_h^k$ be the image under a of $\hat{\mathcal{T}}_h^k$.

Next we discuss the computation of geometric quantities on Γ_h^k. Note first that Γ_h^k is defined *parametrically*, not *implicitly* as is Γ. Thus practical computation of geometric quantities such as normals and curvatures on Γ_h^k may involve somewhat different formulas than does computation of the corresponding quantities on Γ.

Let $\vec{\nu}_h^k$ be the (piecewise smooth) unit normal on Γ_h^k. In order to compute $\vec{\nu}_h^k$ in a practical situation, we let K be a unit simplicial reference element lying in $\mathbb{R}^n$. Let $\hat{T} \in \hat{\mathcal{T}}_h^k$ with $\hat{T} = a_k(\tilde{T})$ where $\tilde{T} \in \tilde{T}_h$, and let $\mathbf{M} : K \to \tilde{T}$ be an affine coordinate transformation with $\mathbf{M}(K) = \tilde{T}$. A typical finite element code allows easy access to the quantities $\hat{a}_{k,x_1}, ..., \hat{a}_{k,x_n}$, where $x_1, ..., x_n$ are the standard Euclidean coordinates on K and $\hat{a}_k = a_k \circ \mathbf{M}$. $\vec{\nu}_h^k$ is then the outward-pointing unit vector that is perpendicular to $\hat{a}_{k,x_1}, ..., \hat{a}_{k,x_n}$. If $n = 2$, we thus have for $x \in K$

$$\vec{\nu}_h^k(\hat{a}_k(x)) = \pm \frac{\hat{a}_{k,x_1}(x) \times \hat{a}_{k,x_2}(x)}{|\hat{a}_{k,x_1}(x) \times \hat{a}_{k,x_2}(x)|}. \tag{2.5}$$

One advantage of employing higher-order approximations to Γ is that in contrast to piecewise linear approximations, such surfaces have naturally defined pointwise curvatures. This information is explicitly needed in the weak (and thus finite element) formulations of various equations. Fix a point $\hat{a}_k(x) \in \Gamma_h^k$, where $x \in K$ with K and $\hat{a}_k$ as above. The second fundamental form with respect to the basis $\{\hat{a}_{k,x_1}, ..., \hat{a}_{k,x_n}\}$ of the tangent space $T_{\hat{a}_k(x)}$ is given by $II = [\hat{a}_{k,x_i x_j} \cdot \vec{\nu}_h^k]$, and the metric tensor is given by $G = [\hat{a}_{k,x_i} \cdot \hat{a}_{k,x_j}]$. The Weingarten map with respect to the basis $\{\hat{a}_{k,x_1}, ..., \hat{a}_{k,x_n}\}$ is then $\mathbf{H}_{tan} = IIG^{-1}$. It is often desirable to express the Weingarten map with respect to the coordinates of the embedding space $\mathbb{R}^{n+1}$ instead of with respect to the basis of the tangent space induced by $\hat{a}_k$. We thus compute

$$\mathbf{H}_h^k = \left[\hat{a}_{k,x_1} .. \hat{a}_{k,x_n}\right] \mathbf{H}_{tan} \mathbf{P}_n \left[\hat{a}_{k,x_1} .. \hat{a}_{k,x_n} \, \vec{\nu}_h^k\right]^{-1},$$

where $\mathbf{P}_n$ is defined by $(x_1, ..., x_n, x_{n+1}) \to (x_1, ..., x_n)$. The principal curvatures and corresponding eigenbasis of the tangent space may be computed from $\mathbf{H}_h^k$. An alternative when $n = 2$ is to apply the formula $\mathbf{H}_h^k = \nabla_{\Gamma_h^k} \vec{\nu}_h^k$ to (2.5).

We now state results concerning the approximation of Γ by Γ_h^k.

Proposition 2.3. *For h small enough, $\tilde{T} \in \tilde{\mathcal{T}}_h$, $\hat{T} \in \hat{\mathcal{T}}_h^k$, and $1 \le i \le k$,*

$$\|d\|_{L_\infty(\Gamma_h^k)} \le \|a - a_k\|_{L_\infty(\Gamma_h)} \le Ch^{k+1}, \tag{2.6}$$

$$\|a - a_k\|_{W_\infty^i(\tilde{T})} \le Ch^{k+1-i}, \tag{2.7}$$

$$\|\vec{\nu} - \vec{\nu}_h^k\|_{L_\infty(\Gamma_h^k)} \le Ch^k, \tag{2.8}$$

$$\|\mathbf{H} \circ a - \mathbf{H}_h^k\|_{L_\infty(\hat{T})} \le Ch^{k-1}. \tag{2.9}$$

The constants C above depend upon the distance function d and its derivatives.

Proof. (2.6) and (2.7) follow directly from the definition of a_k as the Lagrange interpolant of a and the definition of d (cf. [BS02] for standard results concerning finite element interpolation theory). To prove (2.8), consider a point $\hat{x} \in \Gamma_h^k$, where $\hat{x} = a_k(\tilde{x})$ for $\tilde{x} \in \tilde{T} \subset \Gamma_h$. Employing (2.6) and the smoothness of Γ, we have

$$\begin{aligned} |\vec{\nu}(\hat{x}) - \vec{\nu}_h^k(\hat{x})| \le & |\vec{\nu}(a_k(\tilde{x})) - \vec{\nu}(a(\tilde{x}))| + |\vec{\nu}(a(\tilde{x})) - \vec{\nu}_h^k(a_k(\tilde{x}))| \\ \le & C(\Gamma)h^{k+1} + |\vec{\nu}(a(\tilde{x})) - \vec{\nu}_h^k(a_k(\tilde{x}))|. \end{aligned}$$

Assuming without loss of generality that T lies in the $x_1, ..., x_n$-hyperplane, we next note that $\vec{\nu}(a(\tilde{x}))$ is the outward-facing unit vector orthogonal to $a_{x_1}, ..., a_{x_n}$ and $\vec{\nu}_h^k(a_k(\tilde{x}))$ is the outward-facing unit vector orthogonal to $a_{k,x_1}, ..., a_{k,x_n}$. From (2.7) we have $|a_{x_i} - a_{k,x_i}| \le Ch^k$, and it is also not difficult to compute that $|a_{x_i}|$ is bounded from above and below independent of h for $1 \le i \le n$. Using these facts, one may then compute in an elementary fashion that $|\vec{\nu}(a(\tilde{x})) - \vec{\nu}_h^k(a_k(\tilde{x}))| \le Ch^k$, for example by using the Gram-Schmidt orthonormalization algorithm.

(2.9) may be proved in similar fashion after noting that $\|a_{x_i x_j} - a_{k,x_i x_j}\|_{L_\infty(\hat{T})} \le Ch^{k-1}$ for any element $\hat{T} \subset \Gamma_h^k$. □

Remark 2.4. Because $\mathbf{H}_h^k$ involves the second derivatives of a C^0 interpolant, it is only defined elementwise. However, for $k \ge 2$ a pointwise definition of $\mathbf{H}_h^k$ on an element interface may be defined by taking the limit of $\mathbf{H}_h^k$ as the interface is approached from any adjacent element. Stitching these elementwise approximations together yields a global, piecewise continuous curvature approximation with $O(h^{k-1})$ error. In particular, while $\mathbf{H}_h^k$ viewed globally is a distribution with singular jump terms on element interfaces, it is not necessary to take these jump terms into account in order to obtain a convergent pointwise curvature approximation for higher-order discrete surfaces.

2.4. The correspondence between Γ_h, Γ_h^k, and Γ. Our analysis requires a number of relationships between functions defined on Γ and Γ_h^k, as in [Dz88] and [DD07]. In addition, proving approximation results for the parametric finite element spaces S_{hk}^r will require establishing similar relationships between functions defined on Γ_h^k and Γ_h.

We first establish relationships between functions defined on the continuous surface Γ and the discrete surfaces Γ_h^k. Let $v \in H^1(\Gamma)$ and define the extension $v^\ell(x) = v(a(x))$ for $x \in U$. For $v_h \in H^1(\Gamma_h^k)$ we define the lift $\tilde{v}_h \in H^1(\Gamma)$ by $\tilde{v}_h(a(\tilde{x})) = v_h(x)$, $\tilde{x} \in \Gamma_h$. For $v_h \in H^1(\Gamma_h^k)$, we then define the extension $v_h^\ell(x) = \tilde{v}_h(a(x))$ for any $x \in U$. Also, for $\hat{x} \in \Gamma_h^k$ let $\mu_{hk}(\hat{x})$ satisfy $\mu_{hk}(\hat{x})\,\mathrm{d}\sigma_{hk}(\hat{x}) = \mathrm{d}\sigma(a(\hat{x}))$, where $\mathrm{d}\sigma$ and $\mathrm{d}\sigma_{hk}$ are surface measure on Γ and Γ_h^k, respectively.

PROPOSITION 2.5. *Let $x \in \Gamma_h^k$ and $n = 2, 3$. Then*

$$\mu_{hk}(\hat{x}) = \vec{\nu}(\hat{x}) \cdot \vec{\nu}_h^k(\hat{x}) \Pi_{i=1}^n (1 - d(\hat{x})\kappa_i(\hat{x})). \tag{2.10}$$

Remark 2.6. For $x \in U$, $\kappa_i(x) = \frac{\kappa_i(a(x))}{1+d(x)\kappa_i(a(x))}$; cf. [GT98], [DD07].

Proof. (2.10) is proved in [DD07] for $n = 2$ using properties of the cross product, so we sketch a proof for $n = 3$. Let $\hat{T} \subset \mathbb{R}^n$ be a reference simplex. Let also $f = a_k \circ$L: $\hat{T} \to \tilde{T} \subset \Gamma_h^k$, where $\tilde{T} = a_k(\overline{T})$ for $\overline{T} \in \tilde{T}_h$ and $L : \hat{T} \to \overline{T}$ is one of the obvious natural linear transformations. Let f have Jacobian $\mathbf{F} \in \mathbb{R}^{(n+1)\times n}$ with singular values $\sigma_1, .., \sigma_n$ and singular value decomposition $\mathbf{F} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$. Here $\mathbf{U}$ has orthonormal columns $u_1, ..., u_n, \vec{\nu}_h^k$, $\Sigma \in \mathbb{R}^{(n+1)\times n}$, and $\mathbf{V} \in \mathbb{R}^{n\times n}$ is orthogonal.

Let $\mathrm{d}x$ be Lebesgue measure on $\hat{T}$. First we compute $\mathrm{d}\sigma_{hk} = |\Pi_{i=1}^n \sigma_i|\,\mathrm{d}x$ and $\mathrm{d}\sigma = |\det[(\mathbf{P} - d\mathbf{H})\mathbf{F}\ \vec{\nu}]|\,\mathrm{d}x = [\Pi_{i=1}^n (1 - d\kappa_i)]|\det[\mathbf{PF}\ \vec{\nu}]|\,\mathrm{d}x$. But $|\det[\mathbf{PF}\ \vec{\nu}]| = \sqrt{\det \mathbf{F^T PPF}}$. For $n = 2, 3$, a short computation involving the singular value decomposition yields $\sqrt{\det \mathbf{F^T PPF}} = \vec{\nu} \cdot \vec{\nu}_h^k |\Pi_{i=1}^n \sigma_i|$, which completes the proof. ☐

Next we state identities regarding tangential gradients on Γ, Γ_h, and Γ_h^k (cf. [Dz88], [DD07]). For $v_h \in H^1(\Gamma_h^k)$, $v \in H^1(\Gamma)$, and $\hat{x} \in \Gamma_h^k$,

$$\nabla_{\Gamma_h^k} v^\ell(\hat{x}) = [\mathbf{P}_{h,k}(\hat{x})][(\mathbf{I} - d\mathbf{H})(\hat{x})][\mathbf{P}(\hat{x})]\nabla_\Gamma v(a(\hat{x})), \tag{2.11}$$

$$\nabla_\Gamma v_h^\ell(a(\hat{x})) = [(\mathbf{I} - d\mathbf{H})(\hat{x})]^{-1}[\mathbf{I} - \frac{\vec{\nu}_h^k(\hat{x}) \otimes \vec{\nu}(\hat{x})}{\vec{\nu}_h^k(\hat{x}) \cdot \vec{\nu}(\hat{x})}]\nabla_{\Gamma_h^k} v_h(\hat{x}). \tag{2.12}$$

Here $\mathbf{P}_{h,k} = \mathbf{I} - \vec{\nu}_h^k \otimes \vec{\nu}_h^k$ is the projection onto the tangent space of $\Gamma_{h,k}$. Letting

$$\mathbf{A}_\Gamma(a(\hat{x})) = \frac{1}{\mu_{hk}(\hat{x})}\mathbf{P}(\hat{x})[\mathbf{I} - d(\hat{x})\mathbf{H}(\hat{x})]\mathbf{P}_{h,k}(\hat{x})[\mathbf{I} - d(\hat{x})\mathbf{H}(\hat{x})]\mathbf{P}(\hat{x}) \tag{2.13}$$

for $\hat{x} \in \Gamma_h^k$, (2.11) also yields the integral equality

$$\int_{\Gamma_h^k} \nabla_{\Gamma_h^k} u_h \nabla_{\Gamma_h^k} v_h \,\mathrm{d}\sigma_{hk} = \int_\Gamma \mathbf{A}_\Gamma \nabla_\Gamma u_h^\ell \nabla_\Gamma v_h^\ell \,\mathrm{d}\sigma. \tag{2.14}$$

We also shall need to compare Sobolev norms of functions defined on Γ and Γ_h^k. Let $v \in W_p^j(\Gamma)$ with $j \geq 0$ and $1 \leq p \leq \infty$. Then there exist constants C_j depending on j and Γ such that for h small enough,

$$\frac{1}{C_0}\|v\|_{L_p(\Gamma)} \leq \|v^\ell\|_{L_p(\Gamma_h^k)} \leq C_0\|v\|_{L_p(\Gamma)}, \tag{2.15}$$

$$\frac{1}{C_1}\|\nabla_\Gamma v\|_{L_p(\Gamma)} \leq \|\nabla_{\Gamma_h^k} v^\ell\|_{L_p(\Gamma_h^k)} \leq C_1\|\nabla_\Gamma v\|_{L_p(\Gamma)}, \tag{2.16}$$

$$\|D^j_{\Gamma_h^k} v^\ell\|_{L_p(\Gamma_h^k)} \leq C_j \sum_{1\leq m\leq j} \|D^m_\Gamma v\|_{L_p(\Gamma)}. \tag{2.17}$$

The first two inequalities follow from (2.11) and (2.12) along with the equivalence of $\mathrm{d}\sigma$ and $\mathrm{d}\sigma_{hk}$ for h small enough. (2.17) follows from repeated application of (2.11), Proposition 2.3, and the equivalence of $\mathrm{d}\sigma$ and $\mathrm{d}\sigma_{hk}$.

Next we establish analogues of (2.15), (2.16), and (2.17) for functions defined on Γ_h^k and Γ_h. In particular, let $\tilde{T}$ be a triangular face of Γ_h, and let $\hat{T} = a_k(\tilde{T}) \subset \Gamma_h^k$. Let also v be defined and piecewise smooth on Γ_h^k, and for $\tilde{x} \in \tilde{T}$ let $\tilde{v}(\tilde{x}) = v(a_k(\tilde{x}))$. Then there exist positive constants $C_{i,j}$ such that for h small enough,

$$\frac{1}{C_{0,k}}\|v\|_{L_p(\hat{T})} \leq \|\tilde{v}\|_{L_p(\tilde{T})} \leq C_{0,k}\|v\|_{L_p(\hat{T})}, \tag{2.18}$$

$$\frac{1}{C_{1,k}}\|\nabla_{\Gamma_h^k} v\|_{L_p(\hat{T})} \leq \|\nabla_{\Gamma_h}\tilde{v}\|_{L_p(\tilde{T})} \leq C_{1,k}\|\nabla_{\Gamma_h^k} v\|_{L_p(\hat{T})}, \tag{2.19}$$

$$\|D^j_{\Gamma_h}\tilde{v}\|_{L_p(\tilde{T})} \leq C_j \sum_{1\leq m\leq j} \|D^m_{\Gamma_h^k} v\|_{L_p(\hat{T})}. \tag{2.20}$$

We briefly discuss the proof of the above inequalities. Because the transformation $\tilde{x} \to a_k(\tilde{x})$ is the Lagrange interpolant of $\tilde{x} \to a(\tilde{x})$, $\|a_k\|_{W^m_\infty(T)} \leq C\|a\|_{W^m_\infty(T)} \leq C$ for $m \geq 0$ and h small enough. Let $\tilde{\mu}_{hk}$ be defined by $\tilde{\mu}_{hk}(\tilde{x})\,\mathrm{d}\sigma_{h1} = \mathrm{d}\sigma_{hk}(a_k(\tilde{x}))$, $\tilde{x} \in \Gamma_h$. Then $|\mu_{h1} - \tilde{\mu}_{hk}| \leq Ch^k$, so that $\tilde{\mu}_{hk} \approx 1$ for h small enough. These two facts taken together immediately give (2.18), (2.20), and the second inequality in (2.19).

In order to establish the first inequality in (2.19), assume for simplicity that $n = 2$ and T lies in the xy-plane. The general case follows by employing an appropriate coordinate transformation and making the obvious adjustments if $n = 3$. We have

$$\begin{aligned}\nabla_{\Gamma_h}\tilde{v}(\tilde{x}) =& \nabla_{\Gamma_h} v(a_k(\tilde{x}))\\ =& \Big[a_{k,x}\; a_{k,y}\; 0\Big]^T \nabla_{\Gamma_h^k} v(a_k(\tilde{x}))\\ =& \left(\Big[a_{k,x}\; a_{k,y}\; 0\Big]^T + \vec{\nu}_h^k \otimes \vec{\nu}_h^k\right)\nabla_{\Gamma_h^k} v(a_k(\tilde{x})).\end{aligned} \tag{2.21}$$

Let $\mathbf{A} = \Big[a_{k,x}(\tilde{x})\, a_{k,y}(\tilde{x})\, 0\Big]^T + \vec{\nu}_h^k(\tilde{x}) \otimes \vec{\nu}_h^k(\tilde{x})$ and $\mathbf{B} = (\mathbf{I} - d\mathbf{H})(\tilde{x}) = \nabla a + \vec{\nu} \otimes \vec{\nu}$ for $\tilde{x} \in \Gamma_h$, and let $\|\cdot\|_2$ be the matrix 2-norm. We first use the fact that $\nabla a = \mathbf{P} - d\mathbf{H}$ to calculate that $|a_z| = |\nabla a \cdot \vec{\nu}_h^1| = |\nabla a \cdot (\vec{\nu}_h^1 - \vec{\nu})| \leq Ch$. In addition, $|a_{k,x} - a_x| + |a_{k,y} - a_y| \leq Ch^k$. Next we note that since $\mathbf{B}$ is defined on Γ_h and approaches the identity as $dist(\Gamma_h, \Gamma) \to 0$, $\|\mathbf{B}\|_2 + \|\mathbf{B}^{-1}\|_2 \leq C$ for h small enough. Thus employing (2.8), we have (again for h small enough) that

$$\begin{aligned}\|\mathbf{A}^{-1}\|_2 \leq& \|\mathbf{A}^{-1} - \mathbf{B}^{-1}\|_2 + \|\mathbf{B}^{-1}\|_2\\ \leq& \|\mathbf{A}^{-1}\|_2\|\mathbf{B} - \mathbf{A}\|_2\|\mathbf{B}^{-1}\|_2 + C\\ \leq& Ch\|\mathbf{A}^{-1}\|_2 + C \leq C.\end{aligned} \tag{2.22}$$

Multiplying (2.21) through by $\mathbf{A}^{-1}$, inserting (2.22) into (2.21), and employing the equivalence of $\mathrm{d}\sigma_h$ and $\mathrm{d}\sigma_{hk}$ yields the first inequality in (2.19).

2.5. Finite element spaces and approximation theory. We begin by defining a family of Lagrange finite element spaces on Γ_h. Let $\tilde{S}_h^r = \{\tilde{\chi} \in C^0(\Gamma_h) : \tilde{\chi}|_{\tilde{T}} \in \mathbb{P}_r \ \forall \ \tilde{T} \in \tilde{\mathcal{T}}_h\}$, where $r \geq 1$ and $\mathbb{P}_r$ is the set of polynomials in n variables of degree r or less. We next define the family $\hat{S}_{hk}^r$ on Γ_h^k by

$$\hat{S}_{hk}^r = \{\hat{\chi} \in C^0(\Gamma_h^k) : \hat{\chi} = \tilde{\chi} \circ a_k^{-1} \text{ for some } \tilde{\chi} \in \tilde{S}_h^r\}.$$

$\hat{S}_{hk}^r$ is an *isoparametric* finite element space if $k = r$, *subparametric* if $k < r$, and *superparametric* if $k > r$. We finally define the corresponding lifted spaces on Γ,

$$S_h^r = \{\chi \in C^0(\Gamma) : \chi = \tilde{\chi}^\ell \text{ for some } \tilde{\chi} \in \tilde{S}_h^r\}$$

and

$$S_{hk}^r = \{\chi \in C^0(\Gamma) : \chi = \hat{\chi}^\ell \text{ for some } \hat{\chi} \in \hat{S}_{hk}^r\}.$$

Note that because $a \circ a_k \neq a$, $S_{hk}^r \neq S_h^r$.

Next we state results concerning finite element approximation theory. We only consider Lagrange-type interpolants as we only need to approximate functions which are sufficiently smooth (H_2^2) to guarantee the availability of point values for $n \leq 3$. For $v \in H_2^2(\Gamma)$, we define the interpolant $I_h^1 = I_h : C^0(\Gamma) \to S_h^r$ by

$$I_h v = (\tilde{I}_h v^\ell)^\ell,$$

where $\tilde{I}_h : C^0(\Gamma_h) \to \tilde{S}_h^r$ is the standard Lagrange interpolant. We also define the interpolant $\hat{I}_h^k : C^0(\Gamma_h^k) \to \hat{S}_{hk}^r$ by $\hat{I}_h^k v(x) = \tilde{I}_h v(a_k^{-1}(x))$, and

$$I_h^k v = (\hat{I}_h^k v^\ell)^\ell.$$

Note that $I_h \neq I_h^k$ since $a \circ a_k(x) \neq a(x)$ for $x \in \Gamma_h$. This is the case even though the nodal points lying on Γ (and thus nodal values) of the two interpolants are the same.

At several points in our presentation we will consider subdomains $D \subset \Gamma$. Let $D_h = int(\cup_{T \in \mathcal{T}_h, \overline{T} \cap \overline{D} \neq \emptyset} \overline{T})$ and $D_{hk} = int(\cup_{T \in \mathcal{T}_h^k, \overline{T} \cap \overline{D} \neq \emptyset} \overline{T})$. Also, for a given parameter $\gamma \geq h$, we let $D_\gamma = \{x \in \Gamma : dist_\Gamma(x, D) < \gamma\}$.

We shall need the following approximation and superapproximation results.

PROPOSITION 2.7. *Assume that $v \in W_p^{r+1}(\Gamma)$ for some $2 \leq p \leq \infty$, let h be small enough, and let $D \subset \Gamma$. Assume either that $I = I_h$, $\tilde{D}_h = D_h$, and $S^r = S_h^r$; or $I = I_h^k$, $\tilde{D}_h = D_{hk}$, and $S^r = S_{hk}^r$. Then for $i = 0, 1$ and $2 \leq m \leq r + 1$,*

$$|v - Iv|_{W_p^i(D)} \leq C h^{m-i} \|v\|_{W_p^m(\tilde{D}_h)}. \tag{2.23}$$

Let also $\omega \in W_\infty^r(\Gamma)$. Then for $\chi \in S^r$,

$$\begin{aligned} &\|\nabla_\Gamma(\omega\chi - I(\omega\chi))\|_{L_p(D)} \\ &\quad \leq C(h^r \|\chi\|_{L_p(\tilde{D}_h)} \|\omega\|_{W_\infty^{r+1}(\Gamma)} + \|\nabla_\Gamma \chi\|_{L_p(\tilde{D}_h)} \sum_{i=1}^r h^i \|\omega\|_{W_\infty^i(\tilde{D}_h)}. \end{aligned} \tag{2.24}$$

Finally, for any $\chi \in S^r$ and any mesh domain $\tilde{D}_h$,

$$\|\nabla_\Gamma \chi\|_{L_2(\tilde{D}_h)} \leq C h^{-1} \|\chi\|_{L_2(\tilde{D}_h)}. \tag{2.25}$$

All constants above depend on sufficiently high derivatives of the distance function d.

Proof. The proof follows by combining (2.15) through (2.20) with standard estimates for the Lagrange interpolant on Γ_h (cf. [BS02]). For example, if $I = I_h^k$ we may prove (2.24) by letting $\tilde{T}$ be a face of Γ_h and $(a \circ a_k)(\tilde{T}) = T \subset \Gamma$. Let $\tilde{\chi}(x) = \chi((a \circ a_k)(x))$ and $\tilde{\omega}(x) = \omega((a \circ a_k)(x))$ for $x \in \tilde{T}$. (2.15) and (2.19), standard approximation and inverse results on $\tilde{T}$, and (2.17) and (2.20) then yield

$$\begin{aligned}
\|\nabla_\Gamma(\omega\chi - I_h(\omega\chi))\|_{L_p(T)} &\leq C_1 C_{1,k} \|\nabla_{\Gamma_h}(\tilde{\omega}\tilde{\chi} - \tilde{I}_h(\tilde{\omega}\tilde{\chi}))\|_{L_p(\tilde{T})} \\
&\leq C h^r |\tilde{\omega}\tilde{\chi}|_{W_p^{r+1}(\tilde{T})} \leq C h^r \sum_{i=1}^{r+1} |\tilde{\omega}|_{W_\infty^i(\tilde{T})} |\tilde{\chi}|_{W_p^{r+1-i}(\tilde{T})} \\
&\leq C(h^r \|\tilde{\chi}\|_{L_p(\tilde{T})} |\tilde{\omega}|_{W_\infty^{r+1}(\tilde{T})} + \|\nabla_{\Gamma_h}\tilde{\chi}\|_{L_p(\tilde{T})} \sum_{i=1}^{r} h^i |\tilde{\omega}|_{W_\infty^i(\tilde{T})}) \\
&\leq C C_{r+1} C_{r+1,k} [h^r \|\chi\|_{L_p(T)} \|\omega\|_{W_\infty^{r+1}(T)} \\
&\quad + C_1 C_{1,k} \|\nabla_\Gamma \chi\|_{L_p(T)} \sum_{i=1}^{r} h^i \|\omega\|_{W_\infty^i(T)}].
\end{aligned}$$

Summing over $T \cap D \neq \emptyset$ completes the proof of (2.20). The rest of Proposition 2.7 is proved in a similar fashion, with obvious slight simplifications when $I = I_h$. ☐

The proofs of our pointwise estimates also employ a discrete δ-function.

Proposition 2.8. *Let $S^r = S_h^r$ or $S^r = S_{hk}^r$, let $x \in T \subset \Gamma$ with T a surface triangle in either $\mathcal{T}_h$ or $\mathcal{T}_h^k$, and let $\vec{n}$ be a unit vector lying in the tangent plane to Γ at x. Then there exist $\delta_x \in C_0^\infty(T)$ and $\tilde{\delta}_x \in [C_0^\infty(T)]^{n+1}$ such that*

$$\|\delta_x\|_{W_p^j(T)} + \|\tilde{\delta}_x\|_{W_p^j(T)} \leq C h^{-j-n+\frac{n}{p}} \tag{2.26}$$

for $j = 0, 1$ and $1 \leq p \leq \infty$, and for any $\chi \in S^r$,

$$|\chi(x)| \leq C |\int_T \delta_x \chi \,\mathrm{d}\sigma|, \tag{2.27}$$

$$|\nabla_\Gamma \chi(x) \cdot \vec{n}| \leq C |\int_T \chi \nabla_\Gamma \cdot \tilde{\delta}_x \,\mathrm{d}\sigma|. \tag{2.28}$$

Proof. We prove (2.28) when $S^r = S_h^r$; the other cases are similar. Assume $x = a(\tilde{x})$ for $\tilde{x} \in \tilde{T} \in \tilde{\mathcal{T}}_h$, and $T = a(\tilde{T})$. Then employing (2.12), we have

$$\begin{aligned}
|\nabla_\Gamma \chi(x) \cdot \vec{n}| &= |[(\mathbf{I} - d\mathbf{H})(\tilde{x})]^{-1} [\mathbf{I} - \frac{\vec{\nu}_h(\tilde{x}) \otimes \vec{\nu}(\tilde{x})}{\vec{\nu}_h(\tilde{x}) \cdot \vec{\nu}(\tilde{x})}] \nabla_{\Gamma_h} \chi^\ell(\tilde{x}) \cdot \vec{n}| \\
&\leq C |\nabla_{\Gamma_h} \chi^\ell(\tilde{x}) \cdot \vec{n}|.
\end{aligned}$$

Following [SW95], there exists a smooth function $\delta_{\tilde{x}}$ with support in $\tilde{T}$ and not dependent on χ such that $\|\delta_{\tilde{x}}\|_{W_p^k(T)} \leq C h^{-k-n+\frac{n}{p}}$ and $\nabla_{\Gamma_h} \chi^\ell(\tilde{x}) \cdot \vec{n} = \int_{\tilde{T}} \nabla_{\Gamma_h} \chi^\ell \cdot \vec{n} \delta_{\tilde{x}} \,\mathrm{d}\sigma_h$. Employing (2.11) and integrating by parts yields

$$\int_{\tilde{T}} \nabla_{\Gamma_h} \chi^\ell \cdot \vec{n} \delta_{\tilde{x}} \,\mathrm{d}\sigma_h = -\int_T \chi \nabla_\Gamma \cdot ([\mathbf{I} - d\mathbf{H}][\mathbf{P}_h] \vec{n} \frac{1}{\mu_h} \delta_{\tilde{x}}^\ell) \,\mathrm{d}\sigma$$

Setting $\tilde{\delta}_x = \frac{1}{\mu_h} \delta_{\tilde{x}}^\ell [\mathbf{I} - d\mathbf{H}][\mathbf{P}_h] \vec{n}$, we thus have (2.28). The proof of (2.26) is easily accomplished using (2.15) and (2.16). ☐

2.6. Finite element methods. In this section we define two main types of finite element methods. The first type is defined on polynomial approximations of Γ using the spaces $\hat{S}_{hk}^r$. Dziuk's original method in [Dz88] is a special case of this method. The second class of methods involves finite element solutions defined on Γ using the spaces S_h^r and S_{hk}^r.

We first define $\tilde{u}_{hk} \in \hat{S}_{hk}^r$. Let $f_h \in L_2(\Gamma_h^k)$ be an approximation to f^ℓ satisfying $\int_{\Gamma_h^k} f_h \, d\sigma_{hk} = 0$. Then $\tilde{u}_{hk} \in \hat{S}_{hk}^r$ uniquely satisfies $\int_{\Gamma_h^k} \tilde{u}_{hk} \, d\sigma_{hk} = 0$ and

$$\int_{\Gamma_h^k} \nabla_{\Gamma_h^k} \tilde{u}_{hk} \nabla_{\Gamma_h^k} v_h \, d\sigma_{hk} = \int_{\Gamma_h^k} f_h v_h \, d\sigma_{hk} \ \forall \ v_h \in \hat{S}_{hk}^r. \tag{2.29}$$

Dziuk's original method results if we take $k = r = 1$ and $f_h = f^\ell - \frac{1}{|\Gamma_h|} \int_{\Gamma_h} f^\ell \, d\sigma_{h1}$. Using (2.14) while recalling the definition (2.13) of A_Γ and the definition (2.1) of L , we have the perturbed Galerkin orthogonality relationship

$$L(u - \tilde{u}_{hk}^\ell, \chi^\ell) = \int_\Gamma (\mathbf{A}_\Gamma - \mathbf{P}) \nabla_\Gamma \tilde{u}_{hk}^\ell \nabla_\Gamma \chi^\ell \, d\sigma + \int_\Gamma (f - \frac{f_h^\ell}{\mu_{hk}^\ell}) \chi^\ell \, d\sigma, \ \chi \in \hat{S}_{hk}^r.$$

We next define two methods directly on Γ. The first of these methods employs the spaces S_h^r that are defined by lifting polynomial spaces directly from Γ_h. In particular, let $u_{h,\Gamma} \in S_h^r$ satisfy $\int_\Gamma u_{h,\Gamma} \, d\sigma_h = 0$ and

$$\int_\Gamma \nabla_\Gamma u_{h,\Gamma} \nabla_\Gamma v_h \, d\sigma = \int_\Gamma f v_h \, d\sigma \ \forall v_h \in S_h^r. \tag{2.30}$$

$u_{h,\Gamma}$ satisfies the Galerkin orthogonality relationship

$$L(u - u_{h,\Gamma}, \chi) = 0, \ \chi \in S_h^r.$$

So long as one has ready access to the projection a, it is not difficult to program the method (2.30). Indeed, from (2.12) we see that (2.30) may be viewed as a finite element method over Γ_h for an elliptic problem with non-constant elliptic coefficient matrix. (2.30) may thus be regarded as an alternative to our generalized version (2.29) of Dziuk's method which does not involve any geometric error. We emphasize, however, that there are cases where one only has access to a polynomial approximation of Γ, and employing (2.30) is not possible in these cases.

In addition, we let $u_{hk} \in S_{hk}^r$ satisfy $\int_\Gamma u_{hk} = 0$,

$$\int_\Gamma \nabla_\Gamma u_{hk} \nabla_\Gamma v_h \, d\sigma = \int_\Gamma f v_h \, d\sigma \ \forall v_h \in S_{hk}^r. \tag{2.31}$$

u_{hk} satisfies the Galerkin orthogonality relationship

$$L(u - u_{hk}, \chi) = 0, \ \chi \in S_{hk}^r.$$

We employ (2.31) only as a theoretical tool in duality arguments used to prove error bounds in non-energy norms and do not foresee any practical use for it.

3. Abstract error analysis. In this section we prove error estimates for surface finite element methods. Our analysis is carried out under the assumption that the approximation properties proved for the spaces S_h^r and S_{hk}^r in §2.5 hold. We prove our results under general assumptions as we wish our analysis to apply in other situations. In particular, these assumptions will hold if the approximating surfaces Γ_h and Γ_h^k have nodes that lie within $O(h^{k+1})$ of Γ instead of on Γ. It is reasonable to expect that this would be the case when using isoparametric spaces to compute evolving surfaces as in [Dz91], for example.

3.1. Assumptions on the finite element space and solution. We denote by S^r a generic finite element space of degree r. Depending on the error estimate to be proven, we shall require some or all of the following approximation properties:

A1: Basic approximation. We assume that there exists a linear interpolation operator $I : H_2^2(\Gamma) \to S^r$ satisfying (2.23).

A2: Superapproximation. (2.24) holds for any $\chi \in S^r$.

A3: Inverse inequality. (2.25) holds for any $\chi \in S^r$.

A4: Discrete δ function. There exist discrete δ-functions satisfying the properties (2.26), (2.27), and (2.28).

Finally we assume that the finite element approximation $u_h \in S^r$ to u satisfies the perturbed Galerkin orthogonality relationship

$$\int_\Gamma \nabla_\Gamma (u - u_h) \nabla_\Gamma \chi \, \mathrm{d}\sigma = F(\chi) \quad \forall \, \chi \in S^r, \tag{3.1}$$

where F is assumed to be a continuous linear functional on $H^1(\Gamma)/\mathbb{R}$. Here we shall think of F as encoding a geometric error resulting from the discrete approximation of the surface Γ. Thus $F \equiv 0$ for the methods (2.30) and (2.31) defined directly on Γ, while for the method (2.29) defined on polynomial approximations to Γ we have $F(\chi) = \int_\Gamma (\mathbf{A}_\Gamma - \mathbf{I}) \nabla_\Gamma \tilde{u}_{hk}^\ell \nabla_\Gamma \chi \, \mathrm{d}\sigma + \int_\Gamma (f - f_h/\mu_{hk}^\ell) \chi \, \mathrm{d}\sigma$. (The latter version of F is continuous on $H^1(\Gamma)/\mathbb{R}$ because $\int_\Gamma (f - f_h/\mu_{hk}^\ell) \, \mathrm{d}\sigma = 0$.) Such a linear functional F may also be employed to analyze other error sources such as the inexact evaluation of integrals due to numerical quadrature or nonlinearities (cf. the classical work [NS74] and the discussion in [De07]).

3.2. H^1 and L_2 estimates. Here we give local and global H^1 and L_2 estimates. Before doing so, we define the norms

$$|||F|||_{H^{-j}} = \sup_{u \in H^j(\Gamma)/\mathbb{R}, \|u\|_{H^j(\Gamma)/\mathbb{R}} = 1} F(u)$$

and

$$|||F|||_{H^{-1}(D)} = \sup_{u \in H_0^1(D), \|\nabla_\Gamma u\|_{L_2(D)} = 1} F(u), D \subsetneq \Gamma$$

on linear functionals $F : H^1(\Gamma)/\mathbb{R} \to \mathbb{R}$.

THEOREM 3.1. *Assume that $u \in H^1(\Gamma)$ and $u_h \in S^r$ satisfy $L(u - u_h, v_h) = F(v_h)$ for all $v_h \in S^r$, where F is a continuous linear functional on $H^1(\Gamma)/\mathbb{R}$. Then*

$$\|\nabla_\Gamma u_h\|_{L_2(\Gamma)} \le \|\nabla_\Gamma u\|_{L_2(\Gamma)} + C|||F|||_{H^{-1}}, \tag{3.2}$$

$$\|\nabla_\Gamma (u - u_h)\|_{L_2(\Gamma)} \le \min_{\chi \in S^r} \|\nabla_\Gamma (u - \chi)\|_{L_2(\Gamma)} + C|||F|||_{H^{-1}}. \tag{3.3}$$

Let $D \subset \Gamma$ be a subdomain, and let $Kh \le \gamma \le \gamma_\Gamma$ with K sufficiently large and γ_Γ defined as in §2.1. Then if A.1, A.2, and A.3 hold,

$$\begin{aligned} \|\nabla_\Gamma (u - u_h)\|_{L_2(D)} \le & C \min_{\chi \in S^r} (\|\nabla_\Gamma (u - \chi)\|_{L_2(D_\gamma)} + \frac{1}{\gamma} \|u - \chi\|_{L_2(D_\gamma)}) \\ & + \frac{1}{\gamma} \|u - u_h\|_{L_2(D_\gamma)} + |||F|||_{H^{-1}(D_\gamma)}. \end{aligned} \tag{3.4}$$

Finally, let $\overline{u-u_h} = \frac{1}{|\Gamma|}\int_\Gamma (u-u_h)\,\mathrm{d}\sigma$. *Then if A.1 is satisfied,*

$$\|u-u_h-\overline{u-u_h}\|_{L_2(\Gamma)} \le C(h\min_{\chi\in S^r}\|\nabla(u-\chi)\|_{H^1(\Gamma)}+h|||F|||_{H^{-1}}+|||F|||_{H^{-2}}). \quad (3.5)$$

Proof. In order to prove (3.2), we calculate that

$$\begin{aligned}\|\nabla_\Gamma u_h\|^2_{L_2(\Gamma)} =& \int_\Gamma \nabla_\Gamma u \nabla_\Gamma u_h\,\mathrm{d}\sigma - F(u_h)\\ \le& \|\nabla_\Gamma u\|_{L_2(\Gamma)}\|\nabla_\Gamma u_h\|_{L_2(\Gamma)} + |||F|||_{H^{-1}}\|u_h\|_{H^1(\Gamma)/\mathbb{R}}\\ \le&(\|\nabla_\Gamma u\|_{L_2(\Gamma)} + C|||F|||_{H^{-1}})\|\nabla_\Gamma u_h\|_{L_2(\Gamma)},\end{aligned}$$

where C arises from a Poincaré inequality. Dividing through by $\|\nabla_\Gamma u_h\|_{L_2(\Gamma)}$ completes the proof of (3.2). (3.3) may be proved by writing $u-u_h=(u-\chi)-(u_h-\chi)$.

We next prove (3.4). Let $\{D_i\}_{i=1}^N$ be a cover of D consisting of balls of radius $\frac{\gamma}{4}$, and let $D_{i,\gamma/2} = \{x\in\Gamma : dist_\Gamma(x,D_i) < \frac{\gamma}{4}\}$. We may choose the cover so that the balls $D_{i,\gamma/2}$ have finite overlap. Finally let $\omega_i \in C_0^\infty(D_{i,\gamma/2})$ with $\omega_i|_{D_i}\equiv 1$ and $\|\omega_i\|_{W^j_\infty(\Gamma)} \le C\gamma^{-j}$, $0\le j\le r+1$. Such a cutoff function ω exists for $\gamma\le\gamma_\Gamma$. Fixing $\chi\in S^r$, we set $\psi_i=\omega_i^2(\chi-u_h)$ and compute

$$\begin{aligned}\|\nabla_\Gamma(u-u_h)\|^2_{L_2(D)} \le& \sum_{i=1}^N L(\omega_i(u-u_h),\omega_i(u-u_h))\\ =&\sum_{i=1}^N L(u-u_h,\omega_i^2(u-u_h)) + \int_{D_{i,\gamma/2}} |\nabla_\Gamma\omega_i|^2(u-u_h)^2\,\mathrm{d}\sigma\\ \le&\sum_{i=1}^N [L(u-u_h,\omega_i^2(u-\chi)) + L(u-u_h,\psi_i - I\psi_i) + F(I\psi_i)]\\ &+\frac{C}{\gamma^2}\|u-u_h\|^2_{L_2(D_\gamma)}.\end{aligned} \quad (3.6)$$

Next we bound the terms in the last sum in (3.6). For any $1\ge\epsilon>0$,

$$\begin{aligned}L(u-u_h,\omega_i^2(u-\chi)) =& \int_\Gamma \nabla_\Gamma(\omega_i(u-u_h))[\omega_i\nabla_\Gamma(u-\chi)+2(u-\chi)\nabla_\Gamma\omega_i]\,\mathrm{d}\sigma\\ &-\int_\Gamma \omega_i(u-u_h)\nabla_\Gamma\omega_i\nabla_\Gamma(u-\chi)\,\mathrm{d}\sigma - 2\int_\Gamma|\nabla_\Gamma\omega_i|^2(u-u_h)(u-\chi)\,\mathrm{d}\sigma\\ \le&\epsilon\|\nabla_\Gamma(\omega_i(u-u_h))\|^2_{L_2(\Gamma)} + \frac{C}{\epsilon}\|\nabla_\Gamma(u-\chi)\|^2_{L_2(D_{i,\gamma/2})}+\\ &+\frac{C}{\gamma^2\epsilon}(\|u-u_h\|^2_{L_2(D_{i,\gamma/2})} + \|u-\chi\|^2_{L_2(D_{i,\gamma/2})}).\end{aligned} \quad (3.7)$$

Applying (2.24) and (2.25) while recalling that $h\le\gamma$ and $\|\omega_i\|_{W^j_\infty(\Gamma)}\le C\gamma^{-j}$ yields

$$\begin{aligned}&\|\nabla_\Gamma(\psi_i - I_h\psi_i)\|_{L_2(\Gamma)}\\ &\le C\frac{h}{\gamma}(\frac{1}{\gamma}\|\chi-u_h\|_{L_2((D_{i,\gamma/4})_h)} + \|\nabla_\Gamma(\chi-u_h)\|_{L_2((D_{i,\gamma/4})_h)})\\ &\le\frac{C}{\gamma}(\|u-\chi\|_{L_2(D_{i,\gamma/2})} + \|u-u_h\|_{L_2(D_{i,\gamma/2})}).\end{aligned} \quad (3.8)$$

Applying the first line of the previous inequality, we find

$$\begin{aligned} L(u-u_h,\psi_i-I_h\psi_i) \le & C\frac{h}{\gamma}\|\nabla_\Gamma(u-u_h)\|^2_{L_2(D_{i,\gamma/2})} + C\|\nabla_\Gamma(u-\chi)\|^2_{L_2(D_{i,\gamma/2})} \\ & + \frac{C}{\gamma^2}(\|u-u_h\|^2_{L_2(D_{i,\gamma/2})} + \|u-\chi\|^2_{L_2(D_{i,\gamma/2})}). \end{aligned} \tag{3.9}$$

Applying the second line of (3.8) and noting that $\|\nabla_\Gamma \psi_i\|_{L_2(D_{i,\gamma/2})} \le \|\nabla_\Gamma(u-\chi)\|_{L_2(D_{i,\gamma/2})} + \|\nabla_\Gamma(\omega_i(u-u_h))\|_{L_2(D_{i,\gamma/2})} + \frac{1}{\gamma}\|u-u_h\|_{L_2(D_{i,\gamma/2})}$, we finally compute

$$\begin{aligned} \sum_{i=1}^N & F(I\psi_i) = F(\sum_{i=1}^N I\psi_i) \le |||F|||_{H^{-1}(D_{\gamma/2})} \sum_{i=1}^N \|\nabla_\Gamma I\psi_i\|_{L_2(D_{i,\gamma/2})} \\ & \le |||F|||_{H^{-1}(D_{\gamma/2})}[\sum_{i=1}^N \|\nabla_\Gamma(I\psi_i-\psi_i)\|_{L_2(D_{i,\gamma/2})} + \|\nabla_\Gamma \psi_i\|_{L_2(D_{i,\gamma/2})}] \\ & \le \frac{C}{\epsilon}|||F|||^2_{H^{-1}(D_{\gamma/2})} + \frac{C}{\gamma^2}(\|u-\chi\|^2_{L_2(D_{\gamma/2})} + \|u-u_h\|^2_{L_2(D_{\gamma/2})}) \\ & \quad + C\|\nabla_\Gamma(u-\chi)\|^2_{L_2(D_{\gamma/2})} + \epsilon\sum_{i=1}^N \|\nabla_\Gamma(\omega_i(u-u_h))\|^2_{L_2(\Gamma)}. \end{aligned} \tag{3.10}$$

Combining (3.7), (3.9), and (3.10) into (3.6) yields

$$\begin{aligned} \sum_{i=1}^N & \|\nabla_\Gamma(\omega_i(u-u_h))\|^2_{L_2(D_{i,\gamma/2})} \le C(\epsilon)[\frac{1}{\gamma^2}(\|u-\chi\|^2_{L_2(D_{\gamma/2})} \\ & + \|u-u_h\|^2_{L_2(D_{\gamma/2})}) + \|\nabla_\Gamma(u-\chi)\|^2_{L_2(D_{\gamma/2})} + |||F|||^2_{H^{-1}(D_{\gamma/2})}] \\ & + \frac{Ch}{\gamma}\|\nabla_\Gamma(u-u_h)\|^2_{L_2(D_{\gamma/2})} + 2\epsilon\sum_{i=1}^N \|\nabla_\Gamma(\omega_i(u-u_h))\|^2_{L_2(D_{i,\gamma/2})}. \end{aligned} \tag{3.11}$$

The last term in (3.11) may be kicked back by taking $\epsilon = \frac{1}{4}$, yielding

$$\begin{aligned} \|\nabla_\Gamma(u-u_h)\|^2_{L_2(D)} \le & C[\frac{1}{\gamma^2}(\|u-\chi\|^2_{L_2(D_{\gamma/2})} + \|u-u_h\|^2_{L_2(D_{\gamma/2})}) \\ & + \|\nabla_\Gamma(u-\chi)\|^2_{L_2(D_{\gamma/2})} + |||F|||^2_{H^{-1}(D_{\gamma/2})} + \frac{h}{\gamma}\|\nabla_\Gamma(u-u_h)\|^2_{L_2(D_{\gamma/2})}] \end{aligned} \tag{3.12}$$

The term $\frac{h}{\gamma}\|\nabla_\Gamma(u-u_h)\|^2_{L_2(D_{\gamma/2})}$ above may be eliminated by iterating (3.12) with $D_{\gamma/2}$ and D_γ replacing D and $D_{\gamma/2}$, respectively. This results in a term $\frac{h^2}{\gamma^2}\|\nabla_\Gamma(u-\chi)+\nabla_\Gamma(\chi-u_h)\|^2_{L_2(D_\gamma)}$ which may be eliminated by using the triangle inequality and an inverse inequality.

In order to prove (3.5), we first let $z \in H^1(\Gamma)$ solve $L(v,z) = (v, e-\overline{e})_\Gamma$, $\int_\Gamma z\,\mathrm{d}\sigma = 0$, where $e = u-u_h$ and $\overline{e} = \overline{u-u_h}$. Then using (2.23), (2.2), and (3.3) yields

$$\begin{aligned} \|e-\overline{e}\|^2_{L_2(\Gamma)} & = (e-\overline{e}, -\Delta_\Gamma z) = L(e, z-I_h z) + F(I_h z - z) + F(z) \\ & \le C\|\nabla_\Gamma e\|_{L_2(\Gamma)}\|\nabla_\Gamma(z-I_h z)\|_{L_2(\Gamma)} + |||F|||_{H^{-1}}\|z-I_h z\|_{H^1(\Gamma)} \\ & \quad + |||F|||_{H^{-2}}\|z\|_{H^2_2(\Gamma)} \\ & \le C(h\min_{\chi\in S^r}\|\nabla_\Gamma(u-\chi)\|_{L_2(\Gamma)} + h|||F|||_{H^{-1}} + |||F|||_{H^{-2}})\|z\|_{H^2_2(\Gamma)} \\ & \le C(h\min_{\chi\in S^r}\|\nabla_\Gamma(u-\chi)\|_{L_2(\Gamma)} + h|||F|||_{H^{-1}} + |||F|||_{H^{-2}})\|e-\overline{e}\|_{L_2(\Gamma)}. \end{aligned}$$

Dividing through by $\|e-\overline{e}\|_{L_2(\Gamma)}$ completes the proof. □

3.3. Pointwise estimates: Statement of results. In this subsection we state pointwise stability and error estimates. Following [Sch98], let $\sigma_x(y) = \frac{h}{\alpha(x,y)+h}$, where we recall that $\alpha(x,y)$ is the surface distance on Γ. We then define the weighted norm

$$\|u\|_{W_p^j,x,s} = \sum_{0\le|\alpha|\le j} \|\sigma_x^s D^\alpha u\|_{L_p(\Gamma)}.$$

Letting q be the conjugate exponent to p, we define the weighted norm

$$|||F|||_{W_p^{-j},x,s} = \sup_{\|v\|_{W_q^j,x,-s}=1} F(v). \tag{3.13}$$

We shall drop the subscripts x and s in (3.13) when $s=0$.

Theorem 3.2. *Let $0\le s\le r-1$ and $0\le t\le r$, and assume that A1, A2, A3, and A4 all hold. Then for any $x\in\Gamma$,*

$$\begin{aligned}|(u-u_h-&\overline{u-u_h})(x)| \\ &\le C\ell_{h,s}\inf_{\chi\in S^r}(h\|\nabla_\Gamma(u-\chi)\|_{L_\infty,x,s}+\|u-\chi\|_{L_\infty,x,s}) \\ &\quad + C(h\ell_{h,s}|||F|||_{W_\infty^{-1},x,s}+\ell_h|||F|||_{W_\infty^{-2}}),\end{aligned} \tag{3.14}$$

and

$$|\nabla_\Gamma u_h(x)| \le C(\ell_{h,t}\|\nabla_\Gamma u\|_{L_\infty,x,t}+\ell_h|||F|||_{W_\infty^{-1}}), \tag{3.15}$$

$$|\nabla_\Gamma(u-u_h)(x)| \le C(\ell_{h,t}\inf_{\chi\in S^r}\|\nabla_\Gamma(u-\chi)\|_{L_\infty,x,t}+\ell_h|||F|||_{W_\infty^{-1}}). \tag{3.16}$$

Here $\ell_h=\ln\frac{1}{h}$, $\ell_{h,t}=\ell_h$ if $t=r$ and $\ell_{h,t}=1$ otherwise, $\ell_{h,s}=\ell_h$ if $s=r-1$ and $\ell_{h,s}=1$ otherwise.

Taking $s=t=0$ and taking a maximum of (3.14) and (3.16) over Γ yields quasi-optimal L_∞ and W_∞^1 error estimates, modulo analysis of perturbation terms involving F. When $s>0$ (3.14) shows that the pointwise gradient error at x is localized to x in that the weight σ_x^s deemphasizes the approximation error $\nabla(u-\chi)(y)$ by a factor of h^s when $\alpha(x,y)\approx 1$. No localization occurs in errors for function values in the piecewise linear case as $s=r-1=0$ in this case (cf. [De04] for a counterexample). Note that (3.14) and (3.16) are very similar to the results in [Sch98] for domains in $\mathbb{R}^n$. Details peculiar to the fact that we are working on surfaces are hidden in the functional F.

3.4. Proof of Theorem 3.2. We shall prove (3.15) in full detail. The proof of (3.16) follows from (3.15) by writing $\nabla_\Gamma(u-u_h)=\nabla_\Gamma(u-\chi)-\nabla_\Gamma(u_h-\chi)$. The proof of (3.14) is similar but slightly simpler, and we only sketch its proof.

We proceed via a duality argument. Fix a point $x\in\Gamma$, and let $\vec{n}$ be a unit vector lying in the tangent plane to Γ at x. Let $\tilde{\delta}_x$ satisfy the properties (2.26) and (2.28), and let g^x be a discrete Green's function satisfying $L(v,g^x)=(v,\nabla_\Gamma\cdot\tilde{\delta}_x)$ for all $v\in H^1(\Gamma)$ and $\int_\Gamma g^x\,\mathrm{d}\sigma=0$. (Note that $\int_\Gamma\nabla_\Gamma\cdot\tilde{\delta}_x=0$.) Let also $g_h^x\in S^r$ be its finite element approximation satisfying $L(v_h,g^x-g_h^x)=0$ for all $v_h\in S^r$ and $\int_\Gamma g_h^x\,\mathrm{d}\sigma=0$.

Then

$$
\begin{aligned}
|\nabla_\Gamma u_h(x)\cdot\vec{n}| \le & C\int_\Gamma u_h \nabla_\Gamma\cdot\tilde{\delta}_x \,\mathrm{d}\sigma \\
=& L(u_h, g_h^x) = L(u, g_h^x) - F(g_h^x) \\
=& L(u, g_h^x - g^x) + L(u, g^x) - F(g_h^x) \\
\le & \|\nabla_\Gamma u\|_{L_\infty,x,t}\|\nabla_\Gamma(g^x - g_h^x)\|_{L_1(\Gamma),x,-t} + \int_T u\nabla_\Gamma\cdot\tilde{\delta}_x\,\mathrm{d}\sigma \\
& + |||F|||_{W_\infty^{-1}}\|g_h^x\|_{W_1^1(\Gamma)} \\
\le & C\|\nabla_\Gamma u\|_{L_\infty,x,t}(1 + \|\nabla_\Gamma(g^x - g_h^x)\|_{L_1(\Gamma),x,-t}) \\
& + C|||F|||_{W_\infty^{-1}}\|\nabla_\Gamma g_h^x\|_{L_1(\Gamma)},
\end{aligned}
$$

where we have used a Poincaré inequality in the last step.

Similarly, fix $x\in\Gamma$, let $\hat{g}^x$ satisfy $\int_\Gamma \hat{g}^x\,\mathrm{d}\sigma = 0$ and $L(v,\hat{g}^x) = (v, \delta_x - \overline{\delta_x})$ for δ_x satisfying (2.26) and (2.27). Also let $\hat{g}_h^x \in S^r$ satisfy $L(\hat{g}^x - \hat{g}_h^x, \chi) = 0$ for all $\chi \in S^r$ and $\int_\Gamma \hat{g}_h^x\,\mathrm{d}\sigma = 0$. Let also $x \in T$. Then for $\chi \in S^r$,

$$
\begin{aligned}
|(u-u_h)(x) - \overline{u-u_h}| \le & |(u-\chi)(x)| + C|\int_\Gamma (\chi - u_h - \overline{u-u_h})\delta_x\,\mathrm{d}\sigma| \\
\le & C(\|u-\chi\|_{L_\infty(T)} + |L(u-u_h, \hat{g}^x)|) \\
\le & (\|\nabla_\Gamma(u-\chi)\|_{L_\infty,x,s} + |||F|||_{W_\infty^{-1},x,s})\|\hat{g}^x - \hat{g}_h^x\|_{W_1^1,x,-s} \\
& + C\|u-\chi\|_{L_\infty(T)} + |||F|||_{W_\infty^{-2}}\|\hat{g}^x\|_{W_1^2(\Gamma)}.
\end{aligned}
$$

The heart of our proof consists of the following lemma.

Lemma 3.3. *Under the assumptions of §2 and Theorem 3.2,*

$$
\|\nabla_\Gamma(g^x - g_h^x)\|_{L_1,x,-t} \le C\ell_{h,t}, \tag{3.17}
$$
$$
\|\hat{g}^x - \hat{g}_h^x\|_{W_1^1,x,-s} \le Ch\ell_{h,s}, \tag{3.18}
$$
$$
\|\nabla_\Gamma g^x\|_{L_1(\Gamma)} + \|\hat{g}^x\|_{W_1^2(\Gamma)} \le C\ell_h. \tag{3.19}
$$

The proof of (3.16) will be complete once we prove Lemma 3.3.

3.5. Proof of Lemma 3.3. The proof of Lemma 3.3 is similar to that given for domains in $\mathbb{R}^n$ in [Sch98] (though the fact that we here consider an indefinite bilinear form complicates matters slightly). Thus we omit some details from our proof.

Note first that $g^x - g_h^x$ satisfies the error estimates of Theorem 3.1 with $F \equiv 0$. We then decompose Γ into annular subdomains about the point x. For a parameter $M > 0$ which we shall later take to be large enough, we fix $\Gamma_0 = B_{Mh}(x)$ and define $\gamma_j = 2^j Mh$. Let J be the largest integer such that $\gamma_J \le \frac{\gamma_\Gamma}{2}$, where γ_Γ is defined in §2.1. For $0 < j < J$, we define the annuli $\Gamma_j = \{y \in \Gamma : \gamma_{j-1} < \alpha(x,y) < \gamma_j\}$ and then finally define $\Gamma_J = \Gamma \setminus \cup_{0\le j<J}\overline{\Gamma_j}$. Thus $\Gamma = \cup_{0\le j\le J}\Gamma_j$. Also, we let $\Gamma_j' = int(\overline{\Gamma_{j-1}} \cup \overline{\Gamma_j} \cup \overline{\Gamma_{j+1}})$, $\Gamma_j'' = \Gamma_{j-1}' \cup \Gamma_j' \cup \Gamma_{j+1}'$, and $\Gamma_j''' = \Gamma_{j-1}'' \cup \Gamma_j'' \cup \Gamma_{j+1}''$.

We then use (3.4), Hölder's inequality, and (2.23) to find that

$$
\begin{aligned}
&\|\nabla_\Gamma(g^x-g_h^x)\|_{L_1,x,-t}\\
&\quad\le C(M)h^{n/2}\|\nabla_\Gamma(g^x-g_h^x)\|_{L_2(\Gamma_0)}+C\sum_{j=1}^{J}(\frac{\gamma_j}{h})^t\gamma_j^{n/2}\|\nabla_\Gamma(g^x-g_h^x)\|_{L_2(\Gamma_j)}\\
&\quad\le C(M)h^{n/2}[\|\nabla_\Gamma(g^x-g_h^x)\|_{L_2(\Gamma_0)}+h^{-1}\|g^x-g_h^x\|_{L_2(\Gamma_0)}\\
&\qquad+\min_{\chi\in S^r}(\|\nabla_\Gamma(g^x-\chi)\|_{L_2(\Gamma_0)}+h^{-1}\|g^x-\chi\|_{L_2(\Gamma_0)})]\\
&\qquad+\sum_{j=1}^{J}[(\frac{\gamma_j}{h})^t\gamma_j^n h^r\|g^x\|_{W_\infty^{r+1}(\Gamma_{j,h})}+(\frac{\gamma_j}{h})^t\gamma_j^{n/2-1}\|g^x-g_h^x\|_{L_2(\Gamma_j)}].
\end{aligned}
\tag{3.20}
$$

Let $\omega_j\in C_0^\infty(\Gamma_j')$ be a cutoff function satisfying $0\le\omega_j\le 1$ and $\omega_j\equiv 1$ on Γ_j. Let $C_j=\frac{1}{|\Gamma|}\int_{\Gamma_j'}\omega_j^2(g^x-g_h^x)\,\mathrm{d}\sigma$, and let $w\in H^2(\Gamma)$ with $\int_\Gamma w\,\mathrm{d}\sigma=0$ solve

$$L(w,v)=(\omega_j^2(g^x-g_h^x)-C_j,v)\text{ for all }v\in H^1(\Gamma).$$

Using (2.23) and recalling that $\int_\Gamma(g^x-g_h^x)\,\mathrm{d}\sigma=0$, we compute

$$
\begin{aligned}
\|g^x-g_h^x\|_{L_2(\Gamma_j)}^2&\le\|\omega_j(g^x-g_h^x)\|_{L_2(\Gamma)}^2\\
&=(\omega_j^2(g^x-g_h^x)-C_j,g^x-g_h^x)\\
&=L(w,g^x-g_h^x)\\
&=L(w-I_hw,g^x-g_h^x)\\
&\le C(h\|w\|_{H^2(\Gamma_j''')}\|\nabla_\Gamma(g^x-g_h^x)\|_{L_2(\Gamma_j''')}\\
&\quad+h^r\|w\|_{W_\infty^{r+1}(\Gamma\setminus\Gamma_j'')}\|\nabla_\Gamma(g^x-g_h^x)\|_{L_1(\Gamma)}).
\end{aligned}
\tag{3.21}
$$

Noting that $w(y)=\int_\Gamma G^y(z)\omega_j^2(g^x-g_h^x)\,\mathrm{d}\sigma(z)$ since $\int_\Gamma G^y(z)C_j\,\mathrm{d}\sigma(z)=0$, we use (2.4) to calculate that for any multiindex β with $|\beta|\le r+1$ and any $y\in\Gamma\setminus\Gamma_j''$,

$$
\begin{aligned}
D^\beta w(y)&=\int_\Gamma D_y^\beta G^y(z)[\omega_j^2(g^x-g_h^x)]\,\mathrm{d}\sigma(z)\\
&\le\sqrt{|\Gamma_j|}\|\omega_j^2(g^x-g_h^x)\|_{L_2(\Gamma_j')}\|D_y^\beta G^y\|_{L_\infty(\Gamma_j')}\\
&\le C\gamma_j^{n/2}\|\omega_j(g^x-g_h^x)\|_{L_2(\Gamma_j')}\gamma_j^{1-n-r}.
\end{aligned}
\tag{3.22}
$$

Inserting (3.22) into (3.21) and using the regularity estimate (2.2) yields

$$
\begin{aligned}
\|\omega_j(g^x-g_h^x)\|_{L_2(\Gamma)}^2&\le C[h\|\nabla_\Gamma(g^x-g_h^x)\|_{L_2(\Gamma_j''')}\\
&\quad+\gamma_j^{-n/2+1}(\frac{h}{\gamma_j})^r\|\nabla_\Gamma(g^x-g_h^x)\|_{L_1(\Gamma)}]\|\omega_j(g^x-g_h^x)\|_{L_2(\Gamma)},
\end{aligned}
$$

so that

$$
\begin{aligned}
\|g^x-g_h^x\|_{L_2(\Gamma_j)}&\le Ch\|\nabla_\Gamma(g^x-g_h^x)\|_{L_2(\Gamma_j''')}\\
&\quad+\gamma_j^{-n/2+1}(\frac{h}{\gamma_j})^r\|\nabla_\Gamma(g^x-g_h^x)\|_{L_1(\Gamma)}.
\end{aligned}
\tag{3.23}
$$

Recalling (2.26), we next compute that for $y \in \Gamma_{j,h}$ and β with $|\beta| = r+1$,

$$\begin{aligned} D^\beta g^x(y) &= -\int_\Gamma \nabla_{\Gamma,z} D^\beta_y G^y(z) \tilde{\delta}_x(z) \, \mathrm{d}\sigma(z) \\ &\le \|\nabla_\Gamma D^\beta_y G^y\|_{L_\infty(supp(\tilde{\delta}_x))} \|\tilde{\delta}_x\|_{L_1(\Gamma)} \\ &\le C\gamma_j^{-n-r}. \end{aligned} \tag{3.24}$$

Finally, employing (3.3), (3.5), (2.23), (2.2), and (2.26) yields

$$\begin{aligned} C(M)h^{n/2}[&\|\nabla_\Gamma(g^x - g^x_h)\|_{L_2(\Gamma_0)} + h^{-1}\|g^x - g^x_h\|_{L_2(\Gamma_0)} \\ &+ \min_{\chi \in S^r_h}(\|\nabla_\Gamma(g^x - \chi)\|_{L_2(\Gamma_0)} + h^{-1}\|g^x - \chi\|_{L_2(\Gamma_0)}] \\ \le & Ch^{n/2+1}\|\nabla_\Gamma \cdot \tilde{\delta}_x\|_{L_2(\Gamma)} \le C. \end{aligned} \tag{3.25}$$

Inserting (3.23), (3.24), and (3.25) into (3.20), rearranging terms, and finally employing (3.25) yields

$$\begin{aligned} \|\nabla_\Gamma(g^x - g^x_h)\|_{L_1,x,-t} &\le C + C\sum_{j=1}^J (\frac{\gamma_j}{h})^t \gamma_j^{n/2} \|\nabla_\Gamma(g^x - g^x_h)\|_{L_2(\Gamma_j)} \\ &\le C + C\sum_{j=1}^J (\frac{\gamma_j}{h})^t \gamma_j^n h^r \gamma_j^{-r-n} + C\sum_{j=1}^J (\frac{\gamma_j}{h})^t \gamma_j^{n/2} \frac{h}{\gamma_j} \|\nabla_\Gamma(g^x - g^x_h)\|_{L_2(\Gamma_j)} \\ &\quad + C\|\nabla_\Gamma(g^x - g^x_h)\|_{L_1} \sum_{j=1}^J (\frac{\gamma_j}{h})^t (\frac{h}{\gamma_j})^r \\ &\le C + C(1 + \|\nabla_\Gamma(g^x - g^x_h)\|_{L_1(\Gamma)}) \sum_{j=1}^J (\frac{h}{\gamma_j})^{r-t} \\ &\quad + \frac{C}{M} \sum_{j=1}^J (\frac{\gamma_j}{h})^t \gamma_j^{n/2} \|\nabla_\Gamma(g^x - g^x_h)\|_{L_2(\Gamma_j)}. \end{aligned}$$

The last term above may be kicked back (to the last term in the first line) for M large enough. In addition, we note that $\sum_{j=1}^J (\frac{h}{\gamma_j})^{r-t} \le C\ell_{h,t}\frac{1}{M^{r-t}}$. Thus

$$\|\nabla_\Gamma(g^x - g^x_h)\|_{L_1,x,-t} \le C + \frac{C}{M^{r-t}}\ell_{h,t}\|\nabla_\Gamma(g^x - g^x_h)\|_{L_1(\Gamma)}. \tag{3.26}$$

Applying (3.26) with $t = 0$ and taking M large enough to kick back the last term yields

$$\|\nabla_\Gamma(g^x - g^x_h)\|_{L_1} \le C. \tag{3.27}$$

Inserting (3.27) into (3.26) completes the proof of (3.17).

In order to prove the inequality $\|\nabla_\Gamma g^x\|_{L_1(\Gamma)} \le \ell_h$ from (3.19), we first note the easily-proven regularity estimate

$$\|\nabla_\Gamma g^x\|_{L_2(\Gamma)} \le C\|\tilde{\delta}_x\|_{L_2(\Gamma)} \le Ch^{-n/2}.$$

Computing as in (3.24) yields $D^\alpha g^x(y) \le C\alpha(x,y)^{-2}$ for $|\alpha| = 1$ and $\alpha(x,y) \ge 3h$. We thus find that

$$\begin{aligned}\|\nabla_\Gamma g^x\|_{L_1(\Gamma)} \le & Ch^{n/2}\|\nabla_\Gamma g^x\|_{L_2(\Gamma)} + \|\nabla_\Gamma g^x\|_{L_1(\Gamma\setminus B_{3h}(x))} \\ \le & C + \int_{3h}^{C} y^{-1}\,\mathrm{d}y \le C\ell_h.\end{aligned}$$

The proof of (3.18) and the inequality $\|\hat{g}^x\|_{W_1^2(\Gamma)} \le C\ell_h$ is very similar to the corresponding proofs for the appropriate norms of $g^x - g_h^x$ and $\hat{g}^x$ and also to the proofs given in [Sch98], so we only make a couple of notes. First, (3.18) requires us to bound a weighted W_1^1 norm of $\hat{g}^x - \hat{g}_h^x$, not just an L_1 norm of the gradient as in (3.17). However, if we carry out the computation in (3.20) with $\hat{g}^x - \hat{g}_h^x$ and s in place of $g^x - g_h^x$ and t, respectively, then the last line of (3.20) can easily be shown to bound $\|\hat{g}^x - \hat{g}_h^x\|_{L_1,x,-s}$. Secondly, the right hand side $\delta_x - \overline{\delta_x}$ is not locally supported, which requires a modification when performing computations similar to (3.22) and (3.24). In particular, we note that $\hat{g}^x(y) = \int_\Gamma G^y(z)(\delta_x - \overline{\delta_x})\,\mathrm{d}\sigma(z) = \int_\Gamma G^y(z)\delta_x\,\mathrm{d}\sigma(z)$ and then proceed essentially as in (3.24).

□

4. Error analysis of specific methods and numerical results. In this section we apply the abstract error analysis in §3 to the methods (2.29) and (2.30) in §2.6. In the case of the method (2.29) defined on polynomial approximations to Γ, the resulting error bounds consist of a "PDE" or "almost-best-approximation" type term that arises in essentially every finite element approximation, plus a geometric error term arising from the approximation of Γ by Γ_h^k. We also briefly describe numerical experiments that confirm the structure of our H^1 and L_2 estimates.

4.1. Error estimates for FEM on polynomial approximations to Γ. We first state a fundamental geometric error bound which is an extension of a bound found in [Dz88] to higher-order approximations of Γ.

PROPOSITION 4.1.

$$\|\mathbf{A}_\Gamma - \mathbf{P}\|_{L_\infty(\Gamma)} \le Ch^{k+1}. \tag{4.1}$$

Proof. Recalling that $\|d\|_{L_\infty(\Gamma_{h,k})} \le Ch^{k+1}$ and noting from (2.10) that $|1 - \frac{1}{\mu_{hk}}| \le Ch^{k+1} + C|1 - \vec{\nu}\cdot\vec{\nu}_h^k| \le Ch^{k+1} + C|\vec{\nu} - \vec{\nu}_h^k|^2 \le Ch^{k+1}$, we have $|\mathbf{A}_\Gamma - \mathbf{P}| \le |\mathbf{P}\mathbf{P}_{h,k}\mathbf{P} - \mathbf{P}| + Ch^{k+1}$. But $|\mathbf{P}\mathbf{P}_{h,k}\mathbf{P} - \mathbf{P}| = |(\vec{\nu}_h^k - \vec{\nu}\cdot\vec{\nu}_h^k\vec{\nu}) \otimes (\vec{\nu}_h^k - \vec{\nu}\cdot\vec{\nu}_h^k\vec{\nu})| \le Ch^{2k}$, which completes the proof. □

Next we give H^1 and L_2 estimates.

COROLLARY 4.2. *Let $\tilde{u}_{hk}$ satisfy (2.29) with $f_h = \mu_{hk} f^\ell$. Then if $u \in H^{r+1}(\Gamma)$,*

$$\|\nabla_\Gamma(u - \tilde{u}_{hk}^\ell)\|_{L_2(\Gamma)} \le C(h^r\|u\|_{H^{r+1}(\Gamma)} + h^{k+1}\|\nabla_\Gamma u\|_{L_2(\Gamma)}), \tag{4.2}$$

$$\|u - \tilde{u}_{hk}^\ell - \overline{u - \tilde{u}_{hk}^\ell}\|_{L_2(\Gamma)} \le C(h^{r+1}\|u\|_{H^{r+1}(\Gamma)} + h^{k+1}\|\nabla_\Gamma u\|_{L_2(\Gamma)}), \tag{4.3}$$

where C depends on d and its derivatives.

Remark 4.3. The geometric error in the L_2 estimate (3.5) has the form $h|||F|||_{-1} + |||F|||_{-2}$. However, we can not take advantage of the fact that the norm $|||\cdot|||_{-2}$ is weaker than the norm $|||\cdot|||_{-1}$ in order to achieve a higher order of convergence h^{k+2} for the geometric error in our L_2 estimates. Computational experiments in §4 confirm that the geometric error is indeed of order h^{k+1} for both the L_2 and energy errors.

Remark 4.4. It is possible to show that $|\overline{u - \tilde{u}_{hk}^\ell}| = |\overline{\tilde{u}_{hk}^\ell}| \le Ch^{k+1}\|\nabla_\Gamma u\|_{L_2(\Gamma)}$ for h small enough, so that in fact (4.3) holds with $\|u - \tilde{u}_{hk}^\ell\|_{L_2(\Gamma)}$ on the left hand side. We state (4.3) as we do both to maintain consistency with [Dz88] and because we wish to emphasize that (4.3) is sharp with respect to the order of the geometric error.

Proof. Note first that if $f_h = \mu_{hk} f^\ell$, $\tilde{u}_{hk}$ satisfies (3.1) with $F(\chi) = \int_\Gamma (\mathbf{A}_\Gamma - \mathbf{P})\nabla_\Gamma \tilde{u}_{hk}^\ell \nabla_\Gamma \chi \, d\sigma$. Combining (3.2) and (4.1) yields

$$\begin{aligned} |||F|||_{H^{-1}} \le& Ch^{k+1}\|\nabla_\Gamma \tilde{u}_{hk}^\ell\|_{L_2(\Gamma)} \\ \le& Ch^{k+1}(\|\nabla_\Gamma u\|_{L_2(\Gamma)} + C|||F|||_{H^{-1}}). \end{aligned}$$

Taking h small enough to kick back the last term above yields

$$|||F|||_{H^{-1}} \le Ch^{k+1}\|\nabla_\Gamma u\|_{L_2(\Gamma)}, \tag{4.4}$$

which when combined with (3.3) and (2.23) completes the proof of (4.2).

Noting that $|||F|||_{H^{-2}} \le |||F|||_{H^{-1}}$ and then inserting (4.4) into (3.5) while recalling (2.23) completes the proof of (4.3). □

We now give pointwise error estimates.

COROLLARY 4.5. *Let $\tilde{u}_{hk}$ satisfy (2.29) with $f_h = \mu_{hk} f^\ell$. Let also $0 \le s \le r-1$ and $0 \le t \le r$. Then for any $x \in \Gamma$,*

$$\begin{aligned} &|(u - \tilde{u}_{hk}^\ell)(x) - \overline{u - \tilde{u}_{hk}^\ell}| \\ &\le C\ell_{h,s} \inf_{\chi \in S_{hk}^r} (h\|\nabla_\Gamma(u-\chi)\|_{L_\infty,x,s} + \|u-\chi\|_{L_\infty,x,s}) + Ch^{k+1}\ell_h\|\nabla_\Gamma u\|_{L_\infty(\Gamma)}, \end{aligned} \tag{4.5}$$

$$|\nabla_\Gamma(u - \tilde{u}_{hk}^\ell)(x)| \le C(\ell_{h,t} \inf_{\chi \in S_{hk}^r} \|\nabla_\Gamma(u-\chi)\|_{L_\infty,x,t} + h^{k+1}\ell_h\|\nabla_\Gamma u\|_{L_\infty(\Gamma)}). \tag{4.6}$$

Here C depends on d and its derivatives, and ℓ_h, $\ell_{h,t}$, and $\ell_{h,s}$ are defined as in Theorem 3.2.

Proof. We recall that $F(\chi) = \int_\Gamma (\mathbf{A}_\Gamma - \mathbf{P})\nabla_\Gamma \tilde{u}_{hk}^\ell \nabla_\Gamma \chi \, d\sigma$ and then use (3.15) with $t = 0$ and (4.1) to find that for h small enough,

$$\begin{aligned} \|\nabla_\Gamma \tilde{u}_{hk}^\ell\|_{L_\infty(\Gamma)} \le& C(\|\nabla_\Gamma u\|_{L_\infty(\Gamma)} + \ell_h\|\mathbf{A}_\Gamma - \mathbf{P}\|_{L_\infty(\Gamma)}\|\nabla_\Gamma \tilde{u}_{hk}^\ell\|_{L_\infty(\Gamma)}) \\ \le& C(\|\nabla_\Gamma u\|_{L_\infty(\Gamma)} + h^{k+1}\ell_h\|\nabla_\Gamma \tilde{u}_{hk}^\ell\|_{L_\infty(\Gamma)}) \\ \le& C\|\nabla_\Gamma u\|_{L_\infty(\Gamma)}. \end{aligned}$$

Here we have kicked back the last term on the right hand side by taking h sufficiently small. Thus $|||F|||_{W_\infty^{-1},x,s} + |||F|||_{W_\infty^{-2}} \le Ch^{k+1}\|\nabla_\Gamma u\|_{L_\infty(\Gamma)}$, which when inserted into (3.14) and (3.16) yields (4.5) and (4.6), respectively. □

Taking the maximum of (4.5) and (4.6) with $t = s = 0$ leads to standard quasi-optimal pointwise error estimates. In addition, one can easily use (2.23) and elementary manipulations to prove asymptotic error expansion inequalities similar to those given in [Sch98] for domains in $\mathbb{R}^n$.

COROLLARY 4.6. *Under the conditions of Corollary 4.5,*

$$\begin{aligned} \|u - \tilde{u}_{hk}^\ell - \overline{u - \tilde{u}_{hk}^\ell}\|_{L_\infty(\Gamma)} \le C(\tilde{\ell}_h h^{r+1}\|u\|_{W_\infty^{r+1}(\Gamma)} + Ch^{k+1}\ell_h\|\nabla_\Gamma u\|_{L_\infty(\Gamma)}), \\ \|\nabla_\Gamma(u - \tilde{u}_{hk}^\ell)\|_{L_\infty(\Gamma)} \le C(h^r\|u\|_{W_\infty^{r+1}(\Gamma)} + Ch^{k+1}\ell_h\|\nabla_\Gamma u\|_{L_\infty(\Gamma)}), \end{aligned}$$

where $\tilde{\ell}_h = \ell_h$ if $r = 1$ and $\tilde{\ell}_h = 1$ otherwise. In addition for $0 \le s \le r-1$, $0 \le t \le r$, and $x \in \Gamma$,

$$|(u - \tilde{u}_{hk}^\ell)(x) - \overline{u - \tilde{u}_{hk}^\ell}| \le C\ell_{h,s} h^{r+1}[\sum_{1 \le |\beta| \le r+1} |D_\Gamma^\beta u(x)| + \sum_{r+2 \le |\beta| \le r+s} h^{|\beta|-r-1}|D_\Gamma^\beta u(x)| + h^s \|u\|_{W_\infty^{r+1+s}(\Gamma)}],$$

$$|\nabla_\Gamma(u - \tilde{u}_{hk}^\ell)(x)| \le C\ell_{h,r} h^r[\sum_{1 \le |\beta| \le r+1} |D_\Gamma^\beta u(x)| + \sum_{r+2 \le |\beta| \le r+t} h^{|\beta|-r-1}|D_\Gamma^\beta u(x)| + h^t \|u\|_{W_\infty^{r+1+t}(\Gamma)}].$$

4.2. Error estimates for FEM defined on Γ. In order to obtain error estimates for the method (2.30), we simply apply Theorem 3.1 and Theorem 3.2 with $F \equiv 0$ while recalling (2.23).

COROLLARY 4.7. *Let $u_{h,\Gamma}$ defined by (2.30), and assume $u \in H^{r+1}(\Gamma)$. Then*

$$\|\nabla_\Gamma(u - u_{h,\Gamma})\|_{L_2(\Gamma)} \le Ch^r \|u\|_{H^{r+1}(\Gamma)},$$
$$\|u - u_{h,\Gamma}\|_{L_2(\Gamma)} \le Ch^{r+1} \|u\|_{H^{r+1}(\Gamma)}.$$

For $x \in \Gamma$, $0 \le s \le r-1$, and $0 \le t \le r$,

$$|(u - u_{h,\Gamma})(x)| \le C\ell_{h,s} \inf_{\chi \in S_h^r} (h\|\nabla_\Gamma(u - \chi)\|_{L_\infty,x,s} + \|u - \chi\|_{L_\infty,x,s}),$$
$$|\nabla_\Gamma(u - u_{h,\Gamma})(x)| \le C\ell_{h,t} \inf_{\chi \in S_h^r} \|\nabla_\Gamma(u - \chi)\|_{L_\infty,x,t}.$$

Here $\ell_{h,s}$ and $\ell_{h,t}$ are as defined in Theorem 3.2.

4.3. Numerical experiments. In our numerical experiments we let $\Gamma = \{x \in \mathbb{R}^3 : x_1^2 + x_2^2 + \frac{x_3^2}{9} = 1\}$, that is, Γ is an ellipsoid having principal axes of length 1, 1, and 3. Also, we let $u = x_1$. (Note that $\Delta_\Gamma u \not\equiv 0$ on Γ, even though $u(x) = x_1$ is a harmonic function on $\mathbb{R}^3$.) Computations were performed on a sequence of uniformly refined meshes in all cases, with high-order quadrature being employed. We refer to [DD07] for more implementation details, in particular the numerical approximation of a when as in the current case d is not explicitly available. All methods were implemented using the finite element toolbox ALBERTA [SS05].

In Figure 4.1 we display plots of $\|\nabla_\Gamma(u - u_h)\|_{L_2(\Gamma)}$ versus the number of degrees of freedom (DOF), where $u_h = \tilde{u}_{h1}^\ell$, $u_h = \tilde{u}_{h2}^\ell$, and $u_h = u_{h,\Gamma}$ are the finite element approximations defined on a polyhedral approximation to Γ (via (2.29) with $k = 1$), a quadratic approximation to Γ (via (2.29) with $k = 2$), and Γ (via (2.30)), respectively. Optimal-order decrease for $\|\nabla_\Gamma(u - u_h)\|_{L_2(\Gamma)}$ is $DOF^{-r/2}$, so we display logarithmic lines of various slopes for comparison with computed error trends.

The effect of the geometric error is clearly seen. When $k = 1$ (upper left of Figure 4.1), we obtain optimal order convergence when $r = 1$ and $r = 2$ so that $h^{k+1} \le h^r$. Suboptimal convergence is obtained when $r \ge 3$, as expected. When $k = 2$ (upper right) we obtain optimal convergence for $r \le 3$, but not for $r = 4$. Thus (4.2) is sharp with respect to the geometric error $h^{k+1}\|\nabla_\Gamma u\|_{L_2(\Gamma)}$. Finally, in the bottom plot of Figure 4.1 we observe optimal order convergence for all polynomial degrees $r \le 4$

when defining the finite element method directly on Γ via (2.30). We note, however, that our experiments use high-order quadrature, and the quadrature error is likely to be more pronounced when using (2.30) in practical situations as this formulation essentially involves an elliptic problem with a non-constant coefficient matrix.

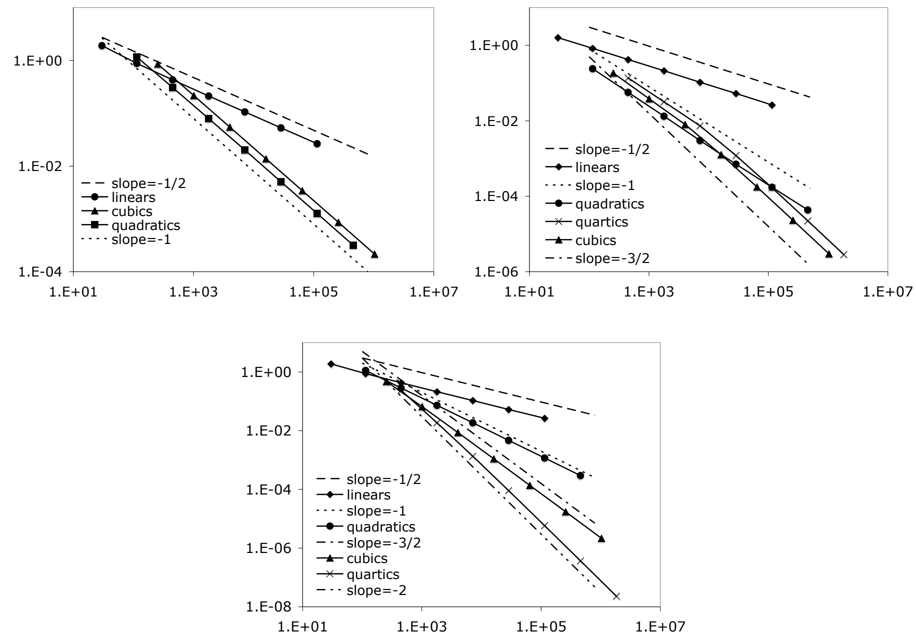

FIG. 4.1. *Plots of* $\|\nabla_\Gamma(u-u_h)\|_{L_2(\Gamma)}$ *vs. the number of degrees of freedom: FEM defined on* Γ_h *(upper left),* Γ_h^2 *(upper right), and* Γ *(bottom).*

Similar plots of the L_2 error on linear and quadratic surface approximations are displayed in Figure 4.2. These plots confirm the sharpness of the error estimate (4.3).

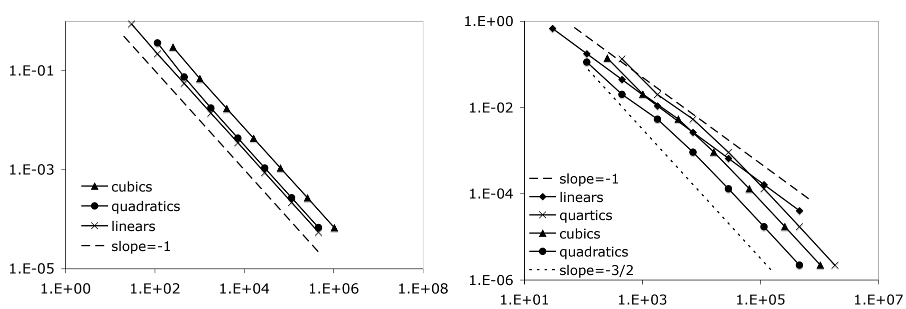

FIG. 4.2. *Plots of* $\|u-u_h-\overline{u-u_h}\|_{L_2(\Gamma)}$ *vs. the number of degrees of freedom: FEM defined on* Γ_h *(left) and* Γ_h^2 *(right).*

5. Extensions. In this section we briefly discuss extensions of our methods and analysis to more general situations.

5.1. More general surface approximations. Our definitions in §2 require that the nodes of the discrete surfaces Γ_h and Γ_h^k lie on Γ. This is a reasonable assumption for stationary problems, but not for geometric evolution problems such as mean curvature flow where the goal is to approximate an unknown surface Γ (cf. [Dz91]). Instead of assuming that the nodes of the discrete surfaces lie on Γ, it is reasonable to assume that they lie within $O(h^{k+1})$ of Γ; cf. the comments at the beginning of §3.

5.2. Surfaces with boundary. Our development may be carried out for surfaces Γ with boundary $\partial\Gamma$ modulo "variational crimes" that arise when $S^r \not\subset H^1(\Gamma)$, just as for domains in $\mathbb{R}^n$. Note that variational crimes do not arise if $\partial\Gamma$ is "curvi-polygonal" in the sense that $a(\partial\Gamma_h) = \partial\Gamma$ (cf. [DD07]). In a few situations, $\partial\Gamma$ may be both smooth and "curvi-polygonal" in this sense (e.g., if Γ is a half-sphere).

5.3. General second-order elliptic PDE. Many applications involve general second-order linear elliptic problems of the form $-\mathrm{div}_\Gamma(\mathcal{D}\nabla_\Gamma \mathrm{u}) + \tilde{\mathrm{b}} \cdot \nabla_\Gamma \mathrm{u} + \mathrm{cu} = \mathrm{f}$. If we make the natural assumption that $\mathcal{D}\vec{\tau}\cdot\vec{\nu} = \vec{b}\cdot\vec{\nu} = 0$ for $\vec{\tau}\cdot\vec{\nu} = 0$ (cf. [DE07b]), then the H^1 and L_2 error estimates of §3 and §4 hold for this problem if the associated bilinear form is coercive and the coefficients sufficiently smooth. In particular, one can show that the geometric error is still of order h^{k+1} in the more general case. Our pointwise estimates hold if a Green's function satisfying the identities and inequalities in Lemma 2.2 exists (note that [Aub82] only considers the Laplace-Beltrami operator).

5.4. C^2 surfaces. In many situations of interest, Γ is not infinitely differentiable. The essential assumption that the orthogonal projection a exists generally requires that Γ be C^2, and situations where Γ is less regular cannot be considered without substantial modification to our methodology. If Γ is merely C^2, the abstract energy and L_2 error estimates of Theorem 3.1 hold verbatim, but the order of the geometric error in Corollary 4.2 is naturally restricted by the smoothness of Γ. We also expect the abstract pointwise estimates of Theorem 3.2 to hold if Γ is only C^2 so long as $s = 0$ and $t \leq 1$. Proving such a statement using our techniques requires the establishment of pointwise estimates for the Green's function as in Lemma 2.2. This can likely be accomplished using an elementary mapping argument, though we have not checked the details.

5.5. Manifolds. The abstract error analysis of §3 relies on two classes of assumptions: those concerning the finite element triangulation and space, and those concerning the underlying partial differential equations. The PDE assumptions employed in §3 hold with slight modification if one considers smooth Riemannian manifolds without boundary instead of smooth surfaces without boundary. Thus if one can construct finite element spaces on manifolds satisfying the assumptions A1 through A4, the results of §3 should hold as well.

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