

Stability of 2^{nd} Hilbert points of canonical curves

Maksym Fedorchuk¹ and David Jensen²

¹Department of Mathematics, Columbia University, 2990 Broadway, New York, NY 10027

²Department of Mathematics, SUNY Stony Brook, Stony Brook, NY 11794

Correspondence to be sent to: mfedorch@math.columbia.edu

We establish GIT semistability of the 2^{nd} Hilbert point of every Gieseker-Petri general canonical curve by a simple geometric argument. As a consequence, we obtain an upper bound on slopes of general families of Gorenstein curves. We also explore the question of what replaces hyperelliptic curves in the GIT quotients of the Hilbert scheme of canonical curves.

Contents

1	Introduction	1
2	Semistability of 2^{nd} Hilbert points	2
3	Degenerations to rational normal surface scrolls	3
4	An answer to the riddle	5
5	A slope inequality après Cornalba and Harris	7

1 Introduction

The log minimal model program for $\overline{\mathcal{M}}_g$, also known as the *Hassett-Keel program*, offers a promising approach to understanding the birational geometry of the moduli space of curves. The goal of this program is to find a functorial interpretation of the log canonical models

$$\overline{\mathcal{M}}_g(\alpha) = \text{Proj} \bigoplus_{m \geq 0} \mathbb{H}^0(\overline{\mathcal{M}}_g, [m(K_{\overline{\mathcal{M}}_g} + \alpha\delta)]).$$

Such an interpretation could then be used to study properties of the rational contraction $\overline{\mathcal{M}}_g \dashrightarrow \overline{\mathcal{M}}_g(\alpha)$ and to obtain structural results about the cone of effective divisors of $\overline{\mathcal{M}}_g$, in particular its Mori chamber decomposition.

Hassett and Hyeon constructed the first two log canonical models of $\overline{\mathcal{M}}_g$ by considering GIT quotients of asymptotically linearized Hilbert schemes of tricanonical and bicanonical curves [13, 12]. It is widely expected that further progress in the Hassett-Keel program will require GIT stability analysis of finite (i.e. non-asymptotic) Hilbert points of bicanonical and canonical curves; see [17, 18, 10, 3]. The case of canonical curves is of particular interest because it should lead to birational contractions of $\overline{\mathcal{M}}_g$ affecting the interior.

Only recently it was shown that finite Hilbert points of general canonical curves are semistable in all genera [2]. In particular, the question of *which* smooth canonical curves have (semi)stable m^{th} Hilbert points is widely open. Here we make partial progress towards answering this question. Our main result is a geometric proof of the semistability of the 2^{nd} Hilbert point of a general canonical curve that leads to a sufficient condition for semistability, something that the previous results lack.

Theorem 1.1. Let C be a Gieseker-Petri general smooth curve of genus $g \geq 4$. Then its canonical embedding $C \subset \mathbb{P}^{g-1}$ has semistable 2^{nd} Hilbert point. □

This result strengthens and complements the proof of generic semistability of 2^{nd} Hilbert points of canonical curves in [2]. Not only do we show that the GIT quotient of the variety of 2^{nd} Hilbert points of canonical curves is non-empty, but also that this GIT quotient parameterizes all curves whose linear systems behave generically. Assuming the expected *stability* of the general canonical curve, this GIT quotient is an interesting projective birational model of \overline{M}_g :

If G is the quotient in question, then the map $f: \overline{M}_g \dashrightarrow G$ is not a local isomorphism along the locus of curves of low Clifford index. As we show in this paper, f is not regular along the hyperelliptic locus $\overline{H}_g \subset \overline{M}_g$ (see Section 4). In addition, f is not regular along the locus $\text{Trig}_g(+)$ of trigonal curves with positive Maroni invariant and contracts the locus $\text{Trig}_g(0)$ of trigonal curves with Maroni invariant 0 to a point (this locus is non-empty only for even g); see Corollary 3.2. We also observe that f is not regular along the bielliptic locus for $g \geq 7$. Finally, in the case $g = 6$, the rational map f contracts both the bielliptic locus (see Proposition 5.2) and the locus of plane quintics (see Corollary 3.5).

In addition to studying the indeterminacy locus of the map $f: \overline{M}_g \dashrightarrow G$, we also examine the indeterminacy locus of its inverse $f^{-1}: G \dashrightarrow \overline{M}_g$. To this end, we show that G parameterizes many different types of singular curves, a large number of which are enumerated in Theorem 3.3. Each of these singularities is predicted to play a role in a functorial interpretation of $\overline{M}_g(\alpha)$; see [1] for precise predictions. As a consequence of our analysis, we discover a class of curves, the A_{2g} -rational curves, which lie in the total transform under f of the hyperelliptic locus \overline{H}_g .

Finally, we include an important application of our semistability result, providing an upper bound on slopes of one-parameter families of Gorenstein curves with a sufficiently general generic fiber.

Theorem 1.2. Let B be a complete curve. If $\mathcal{C} \rightarrow B$ is a flat family of Gorenstein curves with generic fiber a canonically embedded curve whose 2^{nd} Hilbert point is semistable, then the degree λ of the Hodge bundle and the degree δ of the discriminant divisor satisfy the inequality

$$\frac{\delta}{\lambda} \leq 7 + \frac{6}{g}.$$

□

This theorem is an extension of a celebrated result of Cornalba and Harris [7], also independently obtained by Xiao [22], saying that the slope of any generically smooth family of Deligne-Mumford stable curves of genus g is at most $8 + 4/g$.

We prove Theorem 1.2 in Section 5, where we explicate the genericity assumption and give a precise definition of λ and δ . We note in particular that the condition that the generic fiber is GIT semistable implies that it is not hyperelliptic or trigonal with positive Maroni invariant. It has long been expected that lower bounds on the slope of a family of curves should depend on the Clifford index and other geometric properties of the generic fiber; see e.g. [15, 19, 5].

2 Semistability of 2^{nd} Hilbert points

We briefly recall the necessary definitions. Let $C \hookrightarrow \mathbb{P}^{g-1}$ be a canonically embedded smooth curve of genus $g \geq 4$. Using Max Noether's theorem on projective normality of canonical curves [4, p.117], we define the 2^{nd} Hilbert point of $C \hookrightarrow \mathbb{P}^{g-1}$ to be the quotient

$$[\mathbb{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2)) \rightarrow \mathbb{H}^0(C, \mathcal{O}_C(2)) \rightarrow 0] \in \text{Grass}\left(3g-3, \binom{g+1}{2}\right).$$

We denote by $\overline{\text{Hilb}}_g^2$ the closure of the locus of 2^{nd} Hilbert points of canonically embedded curves in the Grassmannian $\text{Grass}(3g-3, \binom{g+1}{2})$ and endow $\overline{\text{Hilb}}_g^2$ with the linearization $\mathcal{O}(1)$ coming from the Plücker embedding of the Grassmannian into $\mathbb{P} \wedge^{3g-3} \mathbb{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2))$. Finally, we set

$$G := \overline{\text{Hilb}}_g^{2, \text{ss}} // \text{SL}(g) = \text{Proj} \bigoplus_{m \geq 0} \mathbb{H}^0(\overline{\text{Hilb}}_g^2, \mathcal{O}(m))^{\text{SL}(g)}$$

to be the resulting GIT quotient. One reason that this construction is of particular interest is that the map

$$f: \overline{M}_g \dashrightarrow G$$

is not an isomorphism on the interior $M_g \subset \overline{M}_g$. More precisely, we show that curves of Clifford index 0 and 1 are outside of the locus where this map is locally an isomorphism. We consider hyperelliptic, trigonal, and bielliptic curves in the later sections of the paper.

We proceed to state the main result of our paper in its greatest generality and to record its most important corollaries.

Theorem 2.1. A canonical curve of genus g not lying on a quadric of rank 3 or less has semistable 2^{nd} Hilbert point. \square

Proof. Our key geometric tool is the $\mathrm{SL}(g)$ -invariant effective divisor $\mathcal{D} \subset \mathrm{Grass}(3g-3, \binom{g+1}{2})$ defined as the locus of $(3g-3)$ -dimensional quotients of $\mathrm{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2))$ whose kernel contains a quadric of rank at most 3. The fact that \mathcal{D} is a divisor follows directly from the fact that the locus of quadrics of rank at most 3 has dimension $3g-3$ in $\mathrm{H}^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2))$. Since $\mathrm{Grass}(3g-3, \binom{g+1}{2})$ is smooth of Picard number 1, the divisor \mathcal{D} is defined by a global section of some power of $\mathcal{O}(1)$. Since $\mathrm{SL}(g)$ has no non-trivial characters, this section is $\mathrm{SL}(g)$ -invariant. It follows that any curve whose Hilbert point is not contained in \mathcal{D} is semistable. \blacksquare

Recall that a complete smooth curve C is said to be *Gieseker-Petri general* if it satisfies the Petri condition that

$$\mu: \mathrm{H}^0(C, L) \otimes \mathrm{H}^0(C, K_C - L) \rightarrow \mathrm{H}^0(C, K_C)$$

is injective for all $L \in \mathrm{Pic}(C)$. That a general curve in M_g is Gieseker-Petri general was proved by Gieseker [11], as well as Eisenbud and Harris [9] using degeneration arguments.

Lemma 2.2. The canonical embedding of a Gieseker-Petri general curve does not lie on a rank 3 quadric. \square

Proof. Suppose a canonically embedded curve lies on a quadric of rank 3 whose vertex is a linear space Λ of dimension $g-3$. The projection away from Λ maps C onto a conic $R \simeq \mathbb{P}^1$ in \mathbb{P}^2 . It follows that there is a decomposition $K_C = 2L + B$. Here, B is an effective divisor with $\mathrm{Supp}(B) = \Lambda \cap C$, and L is a pullback of $\mathcal{O}(1)$ from R . In particular, we have $h^0(C, L) \geq 2$. Let s_0, s_1 be two distinct non-zero global sections of L , and s'_0, s'_1 be the same rational functions considered now as sections of $L + B$. Then

$$\mu(s_0 \otimes s'_1 - s_1 \otimes s'_0) = 0,$$

violating the Petri condition. \blacksquare

Corollary 2.3. Theorem 1.1 holds. \square

Proof. Indeed, Theorem 1.1 is an immediate corollary of Lemma 2.2 and Theorem 2.1. \blacksquare

3 Degenerations to rational normal surface scrolls

Aside from canonical curves, there is another variety of interest in \mathbb{P}^{g-1} with ideal generated by $\binom{g-2}{2}$ quadrics, namely a rational normal surface scroll. Recall that for non-negative integers a and b satisfying $a + b = g - 2$, a rational normal surface scroll $S_{a,b} \subset \mathbb{P}^{g-1}$ is the join of two rational normal curves of degrees a and b whose linear spans do not intersect.

Rational normal surface scrolls are of particular interest to us because the linear system of quadrics containing a smooth trigonal canonical curve $C \subset \mathbb{P}^{g-1}$ cuts out precisely such a surface. Namely, by the geometric Riemann-Roch the g_3^1 's on C are collinear in \mathbb{P}^{g-1} , and the resulting lines sweep out a rational normal surface scroll $S_{a,b}$. The difference $|a - b|$ is classically known as the *Maroni invariant* of C . An important fact is that the ideal of the rational normal surface containing C is generated by the quadrics containing C [4]. It follows that the 2^{nd} Hilbert point of a smooth trigonal canonical curve $C \subset \mathbb{P}^{g-1}$ coincides with the 2^{nd} Hilbert point of the rational normal scroll containing it.

In this section, we show that a rational normal surface scroll $S_{a,b}$ is semistable if and only if $a = b$, i.e. if it is a $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, a)|$ in \mathbb{P}^{2a+1} .

Proposition 3.1. A rational normal surface $S_{a,b}$ has semistable 2^{nd} Hilbert point if and only if $a = b$. \square

Proof. The scroll $S_{a,a}$ in \mathbb{P}^{2a-1} is the image of the homogeneous space $\mathbb{P}^1 \times \mathbb{P}^1$ embedded via the complete linear system $|\mathcal{O}(a-1, 1)|$. The fact that its m^{th} Hilbert point is semistable now follows from Kempf's stability results [14, Corollary 5.3].

Suppose now that $a \neq b$. To see that the scroll $S_{a,b}$ is non-semistable, recall that the ideal of $S_{a,b}$ is generated by the determinants of the 2×2 minors of the following matrix

$$\left(\begin{array}{cccc|cccc} x_0 & x_1 & \cdots & x_{a-1} & y_0 & y_1 & \cdots & y_{b-1} \\ x_1 & x_2 & \cdots & x_a & y_1 & y_2 & \cdots & y_b \end{array} \right)$$

We consider the one-parameter subgroup of $\text{Aut}(S_{a,b}) \subset \text{SL}(g)$ acting with weight $-(b+1)$ on x_i 's and weight $a+1$ on y_i 's. The ideal of $S_{a,b}$ becomes homogeneous with respect to ρ . It follows that every monomial basis of $H^0(S_{a,b}, \mathcal{O}(m))$ has the same ρ -weight. For $m=2$, we compute that the ρ -weight of $H^0(S_{a,b}, \mathcal{O}(2))$ is

$$2(b+1)\binom{a}{2} - 2(a+1)\binom{b}{2} - ab(a-b) = (a-b)(a+b-1) \neq 0.$$

Since the ρ -weight is non-zero, we conclude that $S_{a,b}$ is non-semistable. \blacksquare

As a corollary, we obtain the following two results.

Corollary 3.2. Trigonal curves with positive Maroni invariant have non-semistable 2^{nd} Hilbert points. Trigonal curves with Maroni invariant 0 are strictly semistable and are identified in $G = \overline{\text{Hilb}}_g^{2,ss} // \text{SL}(g)$ with the point corresponding to the balanced rational normal surface scroll in \mathbb{P}^{g-1} . \square

Proof. This follows immediately from Proposition 3.1 and the fact that the 2^{nd} Hilbert point of a trigonal curve of genus g and Maroni invariant m coincides with the 2^{nd} Hilbert point of the scroll $S_{\frac{g+m}{2}-1, \frac{g-m}{2}-1}$. \blacksquare

We note that non-semistability of trigonal curves with a positive Maroni invariant reflects the fact that the locus in \overline{M}_g of trigonal curves contained in an unbalanced scroll is covered by families of slope strictly greater than $7 + \frac{6}{g}$; in particular, when g is odd, Trig_g is covered by families of slope $7 + \frac{20}{3g+1}$ [8].

The second corollary of Proposition 3.1 shows that $G = \overline{\text{Hilb}}_g^{2,ss} // \text{SL}(g)$ parameterizes curves with numerous singularities, as predicted by [1].

Theorem 3.3. Suppose g is even. There exist non-trigonal canonical curves with semistable 2^{nd} Hilbert point and possessing the following classes of singularities:

- (a) All A_k singularities with $k \leq 2g+1$.
- (b) All D_k singularities with $k \leq 2g$.
- (c) If $g = 6m - 2$, the singularity $y^3 = x^{g+2}$ and its deformations.
- (d) If $g = 6m - 4$, the singularity $y^3 = x^{g+1}$ and its deformations.
- (e) If $g = 6m$, the singularity $y^3 = x^{g+1}$ and its deformations.

\square

Proof. Let $g = 2k$. We begin by constructing a curve C , with a desired singularity $p \in C$, in the class $(3, k+1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Next we embed C via the restriction of the complete linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, k-1)|$, which is evidently a canonical linear system on C . The 2^{nd} Hilbert point of the canonical embedding of C will then be the 2^{nd} Hilbert point of the balanced normal scroll $S_{k-1, k-1}$, hence semistable by Proposition 3.1. We can then deform C out of the scroll preserving singularities of C and the semistability of its 2^{nd} Hilbert point.

Construction of the singular curve (C, p) on the scroll:

(a) Consider a smooth rational curve C_1 in the class $(2, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Since $h^0(\mathcal{O}_{C_1}(-1, k-1)) = h^1(\mathcal{O}_{C_1}(-1, k-1)) = 0$, the restriction map $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, k)| \rightarrow |\mathcal{O}_{C_1}(2k+1)|$ is bijective. It follows that for every $p \in C_1$, there exists a unique divisor $C_2 \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, k)|$ such that $(C_1 \cdot C_2)_p = (2k+1)$. Evidently, such a divisor is smooth if p is not a ramification point of the projection $\text{pr}_2: C_1 \rightarrow \mathbb{P}^1$.

It follows that for the general point $p \in C_1$, there is a smooth rational curve $C_2 \in |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, k)|$ such that C_1 and C_2 are maximally tangent at p . Namely, we have $(C_1 \cdot C_2)_p = (2k+1)$. It follows that $C := C_1 \cup C_2$ is a curve of class $(3, k+1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with a unique singularity of type A_{2g+1} . The complete linear system $|\mathcal{O}(1, k-1)|$ embeds $\mathbb{P}^1 \times \mathbb{P}^1$ in \mathbb{P}^{g-1} , mapping C_1 and C_2 to rational normal curves, meeting in a singularity of type A_{2g+1} at p . Thus the image of C under this embedding is an A_{2g+1} -rational curve on $S_{k-1, k-1}$; see Definition 4.1.

(b) A curve with a D_{2g} singularity is obtained by taking a nodal curve C_1 of class $(2, 2)$ and a curve C_2 of class $(1, k-1)$ that is tangent with multiplicity $2k-1$ to one of the branches at the node of C_1 .

(c) If $k = 3m - 1$, then we take three rational curves in the class $(1, m)$, all meeting at a single point where they pairwise intersect with multiplicity $2m$. The resulting singularity is analytically isomorphic to $y^3 = x^{g+2}$.

(d) may be proved analogously to Part (c).

(e) We exhibit an explicit curve in the class $(3, 3m + 1)$ with singularity analytically isomorphic to $y^3 = x^{6m+1}$. Namely, consider

$$(y - x^m)^3 - x^{3m+1}y^3 = 0. \quad (3.1)$$

This curve has a rational parameterization $x = t^3$, $y = t^{3m}/(1 - t^{3m+1})$. Evidently, under this parameterization $x = t^3$ and $y - x^m = t^{6m+1} + \dots$. The claim follows.

Having established the existence of a curve C , with a desired singularity $p \in C$, in the class $(3, k + 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$, we must now show that there exists a non-trigonal semistable curve with the same singularity. To do this, we observe that a general equisingular deformation of C in \mathbb{P}^{g-1} is non-trigonal. More precisely, the deformations of C as a subscheme of \mathbb{P}^{g-1} and the deformations of C as a $(3, k + 1)$ divisor on $\mathbb{P}^1 \times \mathbb{P}^1$ both surject smoothly onto the deformation space of the singularity $p \in C$. Since the dimension of the Hilbert scheme of canonical curves is $(3g - 3) + (g^2 - 2g)$, the dimension of the $\mathrm{SL}(g)$ -orbit of the scroll is $g^2 - 2g - 6$, and the dimension of the linear system $|\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(3, k + 1)|$ is $2g + 7$, we conclude that the general equisingular deformation of (C, p) does not lie on the scroll if and only if $3g - 3 > 2g + 1$, or $g > 4$. This concludes the proof. \blacksquare

In the specific case of $g = 6$, there is another surface of interest – the Veronese surface: If $C \subset \mathbb{P}^5$ is a smooth canonical curve of genus 6 that admits a g_5^2 , then any 5 points in a g_5^2 are coplanar by the geometric Riemann-Roch. It follows that each of the quadrics containing C also contains the conic through these five points. The resulting two-dimensional family of conics sweeps out the Veronese surface in \mathbb{P}^5 . Moreover, the ideal of the Veronese surface is generated by the quadrics containing C .

Proposition 3.4. The Veronese surface in \mathbb{P}^5 has semistable 2^{nd} Hilbert point. \square

Proof. This also follows immediately from [14, Corollary 5.3], as the Veronese surface is simply \mathbb{P}^2 embedded in \mathbb{P}^5 via the complete linear system $|\mathcal{O}_{\mathbb{P}^2}(2)|$. \blacksquare

Corollary 3.5. A canonically embedded plane quintic has semistable 2^{nd} Hilbert point, coinciding with the 2^{nd} Hilbert point of a Veronese surface. \square

4 An answer to the riddle

What is the limit of the canonical model of a smooth curve as it degenerates to a hyperelliptic curve? This is the question that opens a well-known paper of Bayer and Eisenbud [6]. In this section, we aim to show that their answer – a ribbon – is only part of the story. In fact, there is a larger class of curves, the A_{2g} -rational curves, that give a canonical answer to this question, at least from the point of view of GIT for canonical curves.

Definition 4.1. A complete connected reduced curve of genus g with a unique singularity of type A_{2g} ($y^2 = x^{2g+1}$) is called an A_{2g} -rational curve. A complete connected reduced curve of genus g with a unique singularity of type A_{2g+1} ($y^2 = x^{2g+2}$) is called an A_{2g+1} -rational curve. \square

Since the genus of the two singularities A_{2g+1} and A_{2g} both equal g , an A_{2g} -rational curve is necessarily irreducible and its normalization is isomorphic to \mathbb{P}^1 . Similarly, an A_{2g+1} -rational curve necessarily has two irreducible components, each isomorphic to \mathbb{P}^1 . We will denote an A_{2g+1} -rational curve C with the singularity $\hat{\mathcal{O}}_{C,p} \simeq k[[x, y]]/(y^2 - x^{2g+2})$ by (C, p) .

Isomorphism classes of A_{2g+1} -rational curves with a fixed pointed normalization are in bijection with closed points of $\mathbb{G}_m \times \mathbb{G}_a^{g-1}$. Indeed, let the pointed normalization of an A_{2g+1} -rational curve be a disjoint union of two pointed rational curves (\mathbb{P}^1, p_1) and (\mathbb{P}^1, p_2) , where the uniformizer at p_1 is x and at p_2 is y . Then the isomorphism class of a (parameterized) A_{2g+1} -curve is specified by a gluing datum $y \mapsto a_1x + \dots + a_gx^g$, where $a_1 \neq 0$, that defines an isomorphism

$$\mathbb{C}[y]/(y^{g+1}) \rightarrow \mathbb{C}[x]/(x^{g+1})$$

along which the two length $g + 1$ subschemes supported at p_1 and p_2 respectively are glued. We call $(a_1, a_2, \dots, a_g) \in \mathbb{G}_m \times \mathbb{G}_a^{g-1}$ the *crimping*, and refer the reader to [21] for a systematic treatment of crimping for singular curves.

Suppose that C is an A_{2g+1} -rational curve given by the gluing datum $y \mapsto a_1x + \dots + a_gx^g$. Since C is a local complete intersection curve, it admits a dualizing line bundle ω_C . While there are numerous ways to get a handle on this line bundle, we will only consider the one that, to us, is the most explicit. Namely, we use the defining property which says that ω_C is the unique line bundle that restricts to $\mathcal{O}(g - 1)$ on each

irreducible rational component and has g global sections. It follows that we can identify K_C with the triple $(\mathcal{O}(g-1), \mathcal{O}(g-1), \kappa_C)$ where

$$\kappa_C = 1 + k_1x + \cdots + k_gx^g \in (\mathbb{C}[x]/(x^{g+1}))^*$$

is a gluing datum for a line bundle on C . Thus the determination of ω_C reduces to computing κ_C .

Proposition 4.2. The canonical line bundle ω_C is defined by

$$\kappa_C = 1 + k_1x + \cdots + k_{g-1}x^{g-1} + k_gx^g,$$

where $k_g = 0$ and k_i , $1 \leq i \leq g-1$, are (uniquely determined) polynomials in $a_1, (a_1)^{-1}, a_2, \dots, a_g$. \square

Proof. Since $\omega_C|_{C_1} \simeq \mathcal{O}(g-1)$ has exactly g global sections $1, y, \dots, y^{g-1}$, all of them have to lift to global sections of ω_C . This means that each of the elements $\kappa_C, \kappa_C y, \dots, \kappa_C y^{g-1}$ of $\mathbb{C}[x]/(x^{g+1})$ must be a linear combination of $1, x, \dots, x^{g-1}$. From this, we immediately obtain that $k_g = 0$. Next, setting to 0 the coefficient of x^g in

$$\kappa_C y^n = (1 + k_1x + \cdots + k_{g-1}x^{g-1})(a_1x + \cdots + a_gx^g)^n,$$

we obtain

$$a_1^n k_{g-n} + n a_1^{n-1} a_2 k_{g-n-1} + \cdots = 0. \quad (4.1)$$

Setting $n = g-1$, this gives $k_1 = -na_2/a_1$, which determines k_1 uniquely. The assertion for k_2, \dots, k_{g-1} follows by induction by applying (4.1) for $n = g-2, \dots, 1$ repeatedly. \blacksquare

Example 4.3 (see [?, Section 2.3.7]). Up to projectivities, there is a unique canonically embedded A_9 -curve C . It can be defined by the crimping datum $y \mapsto x + x^2 + x^3 + x^4$. A quick computation shows that the gluing datum of ω_C is $\kappa_C = 1 - 3x + 5x^2 - 5x^3$. It follows that C can be defined parametrically by

$$x_0 = \begin{pmatrix} 1 \\ 1 - 3x + 5x^2 - 5x^3 \end{pmatrix}, x_1 = \begin{pmatrix} y \\ x - 2x^2 + 2x^3 \end{pmatrix}, x_2 = \begin{pmatrix} y^2 \\ x^2 - x^3 \end{pmatrix}, x_3 = \begin{pmatrix} y^3 \\ x^3 \end{pmatrix}.$$

\square

Example 4.4. Suppose $g = 2k + 1$. Consider the crimping datum

$$y \mapsto x - tx^{k+2},$$

where $t \neq 0$ is a parameter. One easily computes that $\kappa_C = 1 + tkx^{k+1}$ and that the following is a basis of $H^0(C, \omega_C)$:

$$\omega_i = (x^i + t(k-i)x^{k+1+i}, y^i), \quad i = 0, \dots, k-1, \quad \omega_i = (x^i, y^i), \quad i = k, \dots, 2k.$$

\square

We recall the definition of the balanced ribbon of genus $g = 2k + 1$ from [2]: it is a canonical ribbon obtained by gluing $\text{Spec } \mathbb{C}[u, \epsilon]/(\epsilon^2)$ and $\text{Spec } \mathbb{C}[v, \eta]/(\eta^2)$ via the isomorphism

$$\begin{aligned} u &\mapsto v^{-1} + v^{-k-2}\eta, \\ \epsilon &\mapsto v^{-g-1}\eta \end{aligned}$$

of distinguished open affines $\text{Spec } \mathbb{C}[u, u^{-1}, \epsilon]/(\epsilon^2)$ and $\text{Spec } \mathbb{C}[v, v^{-1}, \eta]/(\eta^2)$.

Lemma 4.5. The flat limit as $t \rightarrow 0$ of the A_{2g+1} -curve in Example 4.4 is the balanced canonical ribbon R of genus $g = 2k + 1$. \square

Proof. Recall from [2, Lemma 3.1] that there is a basis of $H^0(R, \omega_R)$ whose elements can be identified with the following polynomials in u and ϵ (here $\epsilon^2 = 0$):

$$z_i = u^i, \quad 0 \leq i \leq k, \quad z_i = u^i + (i-k)u^{i-k-1}\epsilon, \quad k+1 \leq i \leq 2k.$$

We keep the notation of Example 4.4. Note that if we set $\psi_i := \omega_i/(x^{2k}, y^{2k})$ and $w := 1/x$, then

$$\psi_i = (w^i, y^{-i}), \quad 0 \leq i \leq k, \quad \psi_i = (w^i + (i-k)w^{i-k-1}t, y^{-i}), \quad k+1 \leq i \leq 2k.$$

To prove the lemma, it suffices to show that any quadratic relation among the z_i 's is a flat limit of a quadratic relation among the ψ_i 's as $t \rightarrow 0$. If we evaluate a quadratic relation among z_i 's on ψ_i , we obtain an expression of the form $(f(w)t^2, 0)$, where $\deg f(w) \leq 2k - 2$. It remains to show that $(w^i t, 0)$ can be obtained as a quadratic polynomial in the ψ 's for $0 \leq i \leq 2k - 2$. Indeed, we have

$$\begin{aligned} (w^i t, 0) &= (w^{k+i+1} + (i+1)w^i t, y^{-k-i-1})(1, 1) - (w^{k+i} + iw^{i-1} t, y^{-k-i})(w, y^{-1}) \\ &= \psi_{k+i+1}\psi_0 - \psi_{k+i}\psi_1, \text{ for } 0 \leq i \leq k-1, \end{aligned}$$

and

$$\begin{aligned} (w^i t, 0) &= (w^{2k} + kw^{k-1} t, y^{-2k})(x^{i-k+1}, y^{-i+k-1}) - (w^{2k-1} + iw^{k-2} t, y^{-2k+1})(w^{i-k+2}, y^{-i+k-2}) \\ &= \psi_{2k}\psi_{i-k+1} - \psi_{2k-1}\psi_{i-k+2}, \text{ for } k \leq i \leq 2k-2. \end{aligned}$$

■

We conclude with an observation that the general A_{2g+1} -rational curve is semistable. We would prefer the stronger statement that such a curve is in fact stable, but at present we have no proof.

Proposition 4.6. A general A_{2g+1} -rational curve has semistable 2^{nd} Hilbert point. □

Proof. By the above, the variety of A_{2g+1} -rational curves in \mathbb{P}^{g-1} is irreducible. Thus, it suffices to find a single A_{2g+1} -rational curve with semistable 2^{nd} Hilbert point. When g is even, this is already done by Theorem 3.3 (a). In the case of odd genus, we recall from [2] that the balanced canonical ribbon R has semistable 2^{nd} Hilbert point. Since R deforms flatly to A_{2g+1} -rational curves by Lemma 4.5, we are done. ■

Corollary 4.7. A general A_{2g} -rational curve is semistable. □

Proof. The general A_{2g} -rational curve is a deformation of the general A_{2g+1} -rational curve. The statement now follows from Proposition 4.6. ■

5 A slope inequality après Cornalba and Harris

In this section, we prove Theorem 1.2. To set notation, consider a flat proper family $\pi: \mathcal{C} \rightarrow B$ of Gorenstein curves of arithmetic genus $g \geq 4$. By assumption, the relative dualizing sheaf $\omega := \omega_{\mathcal{C}/B}$ is a line bundle. We set

$$\lambda := c_1(\pi_*\omega), \quad \lambda_2 := c_1(\pi_*\omega^2).$$

After a finite base change, we will assume that λ is divisible by g in $\text{Pic}(B)$ and we let $\tilde{\omega} := \omega(-\lambda/g)$. Then the *normalized Hodge bundle* $\mathbb{E} := \pi_*\tilde{\omega}$ has a trivial determinant, i.e. the transition matrices of \mathbb{E} are given by elements of $\text{SL}(g, \mathcal{O}_B)$.

5.0.1 Line bundles on the moduli stack of Gorenstein curves

Let \mathcal{U}_g be the irreducible component of the stack of all complete canonically polarized Gorenstein curves of arithmetic genus g that parameterizes smoothable curves. Then λ and λ_2 are well-defined line bundles on \mathcal{U}_g . We formally define $\delta := 13\lambda - \lambda_2$. Note that $\overline{\mathcal{M}}_g \subset \mathcal{U}_g$ is an open immersion and that on $\overline{\mathcal{M}}_g$ the line bundle δ has a geometric interpretation as the line bundle associated to the Cartier divisor of nodal curves. Under certain conditions this geometric interpretation can be extended to a larger open of \mathcal{U}_g . To do this, we consider the regular locus $\mathcal{U}_g^{\text{reg}} \subset \mathcal{U}_g$ and define $\Delta := \mathcal{U}_g^{\text{reg}} \setminus \mathcal{M}_g$ to be the locus parameterizing singular curves. Let Δ' be the union of those irreducible components of Δ whose generic points parameterize worse than nodal curves. Then on $\mathcal{U}_g^\circ := \mathcal{U}_g^{\text{reg}} \setminus \Delta'$ the irreducible components of Δ are Cartier divisors whose generic points parameterize nodal curves. By construction, the locus of worse than nodal curves in \mathcal{U}_g° is of codimension at least two. Thus the relation $\mathcal{O}(\Delta) = 13\lambda - \lambda_2$ extends from $\overline{\mathcal{M}}_g$ to \mathcal{U}_g° . We conclude that at least on \mathcal{U}_g° , the formally defined line bundle δ is the associated line bundle of the Cartier divisor $\Delta \subset \mathcal{U}_g^\circ$ parameterizing singular curves.

5.0.2 Slopes of families of Gorenstein curves

Given a family $\mathcal{C} \rightarrow B$ as above, with B a complete curve, we define the *slope* of $\mathcal{C} \rightarrow B$ to be $(\delta \cdot B)/(\lambda \cdot B)$. We proceed to prove Theorem 1.2, which is a generalization of a special case of a well-known result of Cornalba and Harris regarding divisor classes associated to families of Hilbert (semi)stable varieties [7]. In the case of families of curves, the Cornalba-Harris theorem says that the slope of an *arbitrary* generically smooth family of stable curves of genus g is at most $8 + 4/g$ [7, Theorem 1.3]. This result has been proved independently by Xiao [22] for the wider class of fibered algebraic surfaces using a vector bundle argument and more recently by Moriwaki [16] using semistability of a certain vector bundle on $\overline{\mathcal{M}}_g$.

Note that the Cornalba-Harris theorem is sharp: The general family of hyperelliptic curves of genus g has slope precisely $8 + 4/g$, while there exist families of bielliptic curves of slope 8 [5, Theorem 2.1] and there are families of trigonal curves of slope $36(g+1)/(5g+1)$ by [19, 1].

Our proof of Theorem 1.2 follows closely the original argument of Cornalba and Harris, which relied on geometric invariant theory. We include details for completeness. We also note that the Cornalba-Harris GIT approach was recently generalized to more general families by Stoppino [20].

Proof of Theorem 1.2: The key input in Cornalba-Harris method [7] is the *asymptotic* Hilbert semistability of the canonically embedded general fiber of $\mathcal{C} \rightarrow B$. Our assumption that the general fiber C has a semistable 2^{nd} Hilbert point is much stronger than asymptotic semistability and so leads to a stronger inequality. On the other hand, not every smooth canonical curve has a semistable 2^{nd} Hilbert point (see Proposition 5.2), so while our inequality is stronger, we have to settle for a genericity assumption on the generic fiber.

By definition, semistability of C is equivalent to the existence of an $SL(g)$ -invariant polynomial $f \in H^0(\mathbb{P}W, \mathcal{O}(d))$, where $W = \bigwedge^{3g-3} \text{Sym}^2 H^0(C, \omega_C)$, that does not vanish at the point

$$\bigwedge^{3g-3} \text{Sym}^2 H^0(C, \omega_C) \rightarrow \bigwedge^{3g-3} H^0(C, \omega_C^2)$$

of $\mathbb{P}W$. Under the usual identification $H^0(\mathbb{P}W, \mathcal{O}(d)) \simeq \text{Sym}^d W$, the polynomial f corresponds to a section of $\text{Sym}^d W$ that maps to a non-zero section of $\text{Sym}^d \bigwedge^{3g-3} H^0(C, \omega_C^2)$.

Consider now the family $\pi: \mathcal{C} \rightarrow B$ as in the statement of the theorem. Let $\mathbb{E} = \pi_*(\tilde{\omega})$ be the normalized Hodge bundle. Here, $\tilde{\omega} = \omega(-\lambda/g)$ and so $\det \mathbb{E} \simeq \mathcal{O}_B$. Using the $SL(g)$ -invariance of f and the fact that transition matrices of $\text{Sym}^d \bigwedge^{3g-3} \text{Sym}^2 \mathbb{E}$ correspond to an $SL(g)$ coordinate change, we conclude that there exists a section of $\text{Sym}^d \bigwedge^{3g-3} \text{Sym}^2 \mathbb{E}$ that maps to a generically non-vanishing section of $\text{Sym}^d \bigwedge^{3g-3} \pi_*(\tilde{\omega}^2)$. Since $\bigwedge^{3g-3} \pi_*(\tilde{\omega}^2)$ is a line bundle on B , we conclude that

$$c_1 \left(\bigwedge^{3g-3} \pi_*(\tilde{\omega}^2) \right) \geq 0.$$

It follows that $c_1(\pi_*(\tilde{\omega}^2)) \geq 0$.

Since $c_1 \pi_*(\tilde{\omega}^2) = \lambda_2 - 2\lambda(3g-3)/g$, we conclude

$$13\lambda - \delta = \lambda_2 \geq 2(3g-3)\lambda/g,$$

which gives the desired inequality

$$\frac{\delta}{\lambda} \leq 7 + \frac{6}{g}. \quad \blacksquare$$

5.1 Bielliptic curves

As [22, Example 4.3] shows, certain double covers of trivial families of elliptic curves give rise to families of bielliptic curves of genus g and slope 8; for reader's convenience, we recall this construction below. Interestingly, Barja proved that any family of curves with a bielliptic generic fiber has slope at most 8 and those of slope 8 are necessarily double covers of isotrivial families of smooth elliptic curves [5, Theorem 2.1].

Example 5.1 (Bielliptic family of slope 8, cf. [22, Example 4.3]). Let E be a curve of genus one. Consider a constant family $X := E \times B$ and a divisor $D \subset X$ of relative degree $(2g-2)$ over B . Since $K_X = \pi^* K_B$, adjunction gives $K_D - \pi^* K_B = (K_X + D) \cdot D - \pi^* K_B \cdot D = D^2$. Thus the number of branch points of $D \rightarrow B$ is D^2 by Riemann-Hurwitz formula. Consider now the double cover $Y \rightarrow X$ branched over D . The singular fibers

of $Y \rightarrow B$ correspond to branch points of $D \rightarrow B$. Assuming the branch points are simple, we conclude that $\delta_{Y/B} = D^2$. On the other hand, by Mumford's formula

$$12\lambda_{Y/B} - \delta_{Y/B} = \kappa_{Y/B} = 2(\omega_{X/B} + D/2)^2 = D^2/2.$$

It follows that

$$\lambda_{Y/B} = D^2/8 = \delta/8.$$

□

We now contrast the computation of Example 5.1 with Theorem 1.2. Since $8 > 7 + \frac{6}{g}$ for $g \geq 7$, Theorem 1.2 implies that the canonically embedded general bielliptic curve of genus $g \geq 7$ must have a non-semistable 2^{nd} Hilbert point. In fact, we have a more precise result.

Proposition 5.2. The 2^{nd} Hilbert point of a canonically embedded smooth bielliptic curve of genus $g \geq 7$ is non-semistable. The 2^{nd} Hilbert point of a canonically embedded smooth bielliptic curve of genus $g = 6$ is strictly semistable. □

Proof. Consider a genus one curve $E \subset \mathbb{P}^{g-2}$ embedded by a degree $g-1$ complete linear system. There are $\binom{g}{2} - (2g-2) = \binom{g+1}{2} - 3(g-1) - 1$ quadrics containing E . It follows that a projective cone $\text{Cone}(E)$ over E in \mathbb{P}^{g-1} is cut out by one less quadric than a smooth canonical curve. In fact, any smooth quadric section of $\text{Cone}(E)$ is a canonically embedded bielliptic curve of genus g , as can be easily verified using adjunction, and conversely every canonically embedded bielliptic curve lies on such a cone. If $C \in |\mathcal{O}_{\text{Cone}(E)}(2)|$, then there are $\binom{g+1}{2} - 3(g-1) - 1$ quadrics in $H^0(C, \mathcal{I}_C(2))$ that are singular at the vertex of $\text{Cone}(E)$. Suppose the vertex has coordinates $[0 : 0 : \dots : 0 : 1]$. If now ρ is the one-parameter subgroup of $\text{SL}(g)$ acting with weights $(-1, -1, \dots, -1, g-1)$, then the ρ -weight of any monomial basis of $H^0(C, \mathcal{I}_C(2))$ is at most

$$-2 \left(\binom{g+1}{2} - 3(g-1) - 1 \right) + 2(g-1) = -g^2 + 7g - 6 = -(g-1)(g-6).$$

Thus a bielliptic curve has non-stable 2^{nd} Hilbert point for all $g \geq 6$, and non-semistable 2^{nd} Hilbert point for all $g \geq 7$.

It remains to establish the semistability of a canonically embedded smooth bielliptic curve of genus 6. By above, every such curve degenerates isotrivially to a double hyperplane section of a cone over a genus one curve of degree 5 in \mathbb{P}^4 . The semistability of this non-reduced curve follows from Kempf's results [14]. ■

Acknowledgement

The first author would like to thank Aise Johan de Jong for several discussions that gave an impetus to this paper and Anand Deopurkar for many fruitful discussions about ribbons, canonical curves, and slopes of sweeping families of trigonal loci. The second author would like to thank Sean Keel for several helpful conversations during the initial stages of this project.

References

- [1] Jarod Alper, Maksym Fedorchuk, and David Smyth. Singularities with \mathbb{G}_m -action and the log minimal model program for \overline{M}_g , 2010. [arXiv:1010.3751v2](#) [math.AG].
- [2] Jarod Alper, Maksym Fedorchuk, and David Smyth. Finite Hilbert stability of (bi)canonical curves, 2011. to appear in *Invent. Math.*, [arXiv:1109.4986](#) [math.AG].
- [3] Jarod Alper and Donghoon Hyeon. GIT constructions of log canonical models of M_g , 2011. *Proceedings of the UGA Conference on Moduli and Vector Bundles*, to appear. [arXiv:1109.2173](#) [math.AG].
- [4] E. Arbarello, M. Cornalba, P. A. Griffiths, and J. Harris. *Geometry of algebraic curves. Vol. I*, volume 267 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, New York, 1985.
- [5] Miguel A. Barja. On the slope of bielliptic fibrations. *Proc. Amer. Math. Soc.*, 129(7):1899–1906 (electronic), 2001.

- [6] Dave Bayer and David Eisenbud. Ribbons and their canonical embeddings. *Trans. Amer. Math. Soc.*, 347(3):719–756, 1995.
- [7] Maurizio Cornalba and Joe Harris. Divisor classes associated to families of stable varieties, with applications to the moduli space of curves. *Ann. Sci. École Norm. Sup. (4)*, 21(3):455–475, 1988.
- [8] Anand Deopurkar and Anand Patel. Slope bounds for sweeping families of trigonal curves, 2011. Preprint.
- [9] David Eisenbud and Joe Harris. A simpler proof of the Gieseker-Petri theorem on special divisors. *Invent. Math.*, 74(2):269–280, 1983.
- [10] Maksym Fedorchuk and David Smyth. Alternate compactifications of moduli spaces of curves, 2010. To appear in the *Handbook of Moduli*, edited by G. Farkas and I. Morrison. [arXiv:1012.0329v2](https://arxiv.org/abs/1012.0329v2) [math.AG].
- [11] David Gieseker. Stable curves and special divisors: Petri’s conjecture. *Invent. Math.*, 66(2):251–275, 1982.
- [12] Brendan Hassett and Donghoon Hyeon. Log minimal model program for the moduli space of stable curves: The first flip, 2008. [arXiv:0806.3444](https://arxiv.org/abs/0806.3444) [math.AG].
- [13] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.
- [14] George R. Kempf. Instability in invariant theory. *Ann. of Math. (2)*, 108(2):299–316, 1978.
- [15] Kazuhiro Konno. Clifford index and the slope of fibered surfaces. *J. Algebraic Geom.*, 8(2):207–220, 1999.
- [16] Atsushi Moriwaki. Relative Bogomolov’s inequality and the cone of positive divisors on the moduli space of stable curves. *J. Amer. Math. Soc.*, 11(3):569–600, 1998.
- [17] Ian Morrison. GIT constructions of moduli spaces of stable curves and maps. In *Surveys in differential geometry. Vol. XIV. Geometry of Riemann surfaces and their moduli spaces*, volume 14 of *Surv. Differ. Geom.*, pages 315–369. Int. Press, Somerville, MA, 2009.
- [18] Ian Morrison and David Swinarski. Gröbner techniques for low-degree Hilbert stability. *Exp. Math.*, 20(1):34–56, 2011.
- [19] Zvezdelina E. Stankova-Frenkel. Moduli of trigonal curves. *J. Algebraic Geom.*, 9(4):607–662, 2000.
- [20] Lidia Stoppino. Slope inequalities for fibered surfaces via GIT. *Osaka J. Math.*, 45(4):1027–1041, 2008.
- [21] Frederick van der Wyck. Moduli of singular curves and crimping, 2010. Ph.D. thesis, Harvard University.
- [22] Gang Xiao. Fibered algebraic surfaces with low slope. *Math. Ann.*, 276(3):449–466, 1987.