Carl Friedrich Gauss

(1777–1855)

HIS LIFE AND WORK

In 1940, the eminent British mathematician G. H. Hardy wrote:

> *317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but because it is so, because mathematical reality is built that way.*

uld this attitude toward mathematics explain why Carl Friedrich Gauss, unques- nably the greatest mathematician of all time, withheld from publication his work non-Euclidean geometry while seeking an empirical verification he would never nd? Perhaps, but we will never know with certainty. What we do know with cer- nty is the enormity of Gauss's mathematical gifts and accomplishments.

Gauss's talents were obvious as soon as he stepped into a classroom at the age seven. When the class began to be unruly, the teacher, J. G. Büttner, assigned m the task of adding up all of the integers from 1 to 100. As his classmates uggled to fit their calculations on their individual slates, Gauss wrote down the wer immediately: 5,050. As soon as the problem was stated, Gauss recognized t the set of integers from 1 to 100 was identical to 50 pairs of integers each ding up to 101: ({1,100}, {2,99}, . . . , {50,51}).

Herr Büttner approached Gauss's parents to persuade them to let their son stay after school for special math instruction. Gauss's parents were at first skeptical. They had recognized their son's calculating ability when, at the age of three, he corrected a mistake his father made in paying out wages to men who worked him. However, the elder Gausses had very limited horizons. Gauss's father, Gebhard, born in 1744, was a gardener, laborer, and foreman who came from a long line of poor, unlettered workmen who had had little success escaping their peasant roots to the lower middle class. His mother, born Dorothea Benze in 1743, was a maid before becoming Gebhard's second wife in 1776. Their only son Carl Friedrich was born a year later in the city of Brunswick, the capital of a Duchy of the same name.

In one respect, Gauss was very fortunate that his parents had such limited horizons and did not abuse his talents. Otherwise, they might have taken him on tour and exhibited him as a calculating prodigy, in much the same way that Mozart's father had taken young Wolfgang on tour as a musical prodigy. By his early teens, Gauss had worked out two methods to compute square roots to as many as fifty decimal places. He is reputed to have read tables of logarithms and found slight errors many places to the right of the decimal point. This computational facility would serve Gauss well later in his life.

Fortunately, he was to have as close to a regular education as possible for someone of his gifts. At the age of eleven, Gauss entered the local Gymnasium and received a thorough education in the classics. His real math education came in one-on-one instruction after school and on his own as he devoured the contents of works like Newton's *Principia* and Bernoulli's *Ars conjectandi* in his spare time. Gauss achieved such a distinguished record at the Gymnasium that at the age of fifteen, Carl Wilhelm Ferdinand, Duke of Brunswick, gave Gauss a stipend to continue his education at the Brunswick Collegium Carolinum.

Gauss entered the Collegium already possessed of a scientific and classical education worthy of a graduate. Three years later, he left the Collegium in 1795 having done enough mathematics to fill a career. During these years, Gauss gave the first proposals for approximations for $\pi(n)$, the function counting the number of prime numbers less than n. Gauss first proposed the following:

$$\pi(n) \sim \frac{n}{\ln n}.$$

($\ln n$ is the natural logarithm of n) and then refined this to $\pi(n) = \mathrm{Li}(n)$, where

$$\mathrm{Li}\,(n) = \int_2^n \frac{dx}{\ln x}$$

Ever the calculator, Gauss computed the number of primes and tested the formula up to 3,000,000.

Gauss sped through the curriculum at the Collegium and enrolled at the University of Göttingen, sixty miles from Brunswick, rather than the Duchy's official university in nearby Helmstedt, most likely due to Göttingen's superior mathematics library. Surprisingly, Göttingen's records show that Gauss borrowed far more books in the humanities than in mathematics. Gauss's notebooks tell us that he had a much higher regard for his classics professor than for his mathematics professor. However, mathematical accomplishment quickly won out.

The Greeks had known that regular polygons with 3 or 5 or 15 sides could be constructed with a straight edge and compass. So could any regular polygon having a number of sides that was a power of 2 times 3, 5, or 15, and that is where the boundary of the field stood for two millennia until 1796 when Gauss discovered that a seventeen-sided regular polygon could also be constructed with the classical geometrical tools: the straight edge and compass. He quickly generalized his result to any regular polygon with a number of sides that is a product of a power of 2 and any number of Fermat primes. A Fermat prime is a prime number of the form $2^N + 1$, where N is itself a power of 2. Fermat (1601–1665) thought that all numbers of the form $2^N + 1$, where N is a power of 2, are prime numbers, and it is easy to see that 3, 5, 17, and 257 have this property. With only a little bit of brute force you can demonstrate, as Fermat must have done, that 65,537 $(= 2^{16} + 1)$ is a prime. The next candidate Fermat prime is 4,294,967,297 $(= 2^{32} + 1)$. We cannot fault Fermat for not finding 641 as its smallest prime divisor. It took a century for the great mathematician Leonhard Euler to find that. In his notebooks, Gauss speculated that there are no other Fermat primes. To date, none have been found to exist.

Gauss was so overjoyed at this result that it convinced him to pursue a career in mathematics. After two years at Göttingen, he realized that no one on the faculty could really be of any assistance to him, so he went home to Brunswick to write his doctoral dissertation. For his topic he chose the fundamental theorem of algebra, that every polynomial equation of degree n with complex coefficients has exactly n roots in the complex numbers. His dissertation was the first of what would be four proofs throughout his career.

Freed of the need to write a set piece, Gauss turned his attention to number theory. Number theory goes back to the Greeks with Euclid's proofs of the infinity of prime numbers and the form of even perfect numbers being two of the earliest results in the field. From time to time, new results were added or new conjectures made. In the seventeenth century, the French mathematician Pierre de Fermat, a contemporary of Descartes, made his famous conjecture that the equation $x^n + y^n = z^n$ has no nontrivial solutions in integers for $n > 2$. The Arabs made same progress for