BIRATIONAL CONTRACTIONS OF $\overline{M}_{3,1}$ AND $\overline{M}_{4,1}$

DAVID JENSEN

Abstract. We study the birational geometry of $\overline{M}_{3,1}$ and $\overline{M}_{4,1}$. In particular, we pose a pointed analogue of the Slope Conjecture and prove it in these low-genus cases. Using variation of GIT, we construct birational contractions of these spaces in which certain divisors of interest – the pointed Brill-Noether divisors – are contracted. As a consequence, we see that these pointed Brill-Noether divisors generate extremal rays of the effective cones for these spaces.

1. Introduction

The moduli spaces of curves are some of the most studied objects in algebraic geometry. In recent years, a great deal of progress has been made on understanding the birational geometry of these spaces. Examples include the work of Hassett and Hyeon on the minimal model program for $\overline{M}_g$ [HH09a] [HH09b] and the discovery by Farkas of previously unknown effective divisors on $\overline{M}_g$ [Far09]. Nevertheless, many fundamental questions remain open.

Many of these questions can be stated in terms of the cone of effective divisors $\overline{NE}^1(\overline{M}_g)$. Among the first to study this cone were Eisenbud, Harris and Mumford in a series of papers proving that $\overline{M}_g$ is of general type for $g \geq 24$ [HM82] [EH87]. A key element of these proofs is the computation of the class of certain divisors on $\overline{M}_g$. The original paper of Harris and Mumford focused on the k-gonal divisor in $\overline{M}_{2k-1}$, a specific case of the more general class of Brill-Noether divisors. In their argument, they use this calculation to show that the canonical class can be written as an effective sum of a Brill-Noether divisor, boundary divisors, and an ample divisor, and hence lies in the interior of $\overline{NE}^1(\overline{M}_g)$. The search for effective divisors with this property eventually led to the Harris-Morrison Slope Conjecture.

In their work, Harris and Eisenbud discovered that all of the Brill-Noether divisors lie on a single ray in $\overline{NE}^1(\overline{M}_g)$. One consequence of the Slope Conjecture would be that this ray is extremal. The Slope Conjecture has recently been proven false in [FP05] and subsequently in [Far09], but the statement is known to hold for certain small values of g. In several of these cases, the statement can be proved by use of the

Contraction Theorem, which states that the set of exceptional divisors of a birational contraction $X \dashrightarrow Y$ span a simplicial face of $\overline{NE}^1(X)$ (see [Rul01]). In other words, the Slope Conjecture has been shown to hold for small values of g by constructing explicit birational models for the moduli space in which the Brill-Noether divisor is contracted. Moreover, these models arise naturally as geometric invariant theory quotients.

The purpose of this paper is to carry out a pointed analogue of the discussion above in some low genus cases. In [Log03], Logan introduced the notion of **pointed Brill-Noether divisors**.

Definition 1. *Let $Z = (a_0, \dots, a_r)$ be an increasing sequence of non-negative integers with $\alpha = \sum_{i=0}^{r}(a_i - i)$. Let $BN^r_{d,Z}$ be the closure of the locus of pointed curves $(p, C) \in M_{g,1}$ possessing a g^r_d on C with vanishing sequence Z at p. When $g + 1 = (r + 1)(g - d + r) + \alpha$, this is a divisor in $\overline{M}_{g,1}$, called a* ***pointed Brill-Noether divisor****.*

Logan's original motivation was to prove a pointed version of the Harris-Mumford general type result. In this setting, it is natural to consider an analogue of the Slope Conjecture:

Question 1. *Is there an extremal ray of $\overline{NE}^1(\overline{M}_{g,1})$ generated by a pointed Brill-Noether divisor?*

We consider this question in certain low-genus cases. When $g = 2$, this question was answered in the affirmative by Rulla [Rul01]. He shows that the Weierstrass divisor $BN^1_{2,(0,2)}$ generates an extremal ray of $\overline{NE}^1(\overline{M}_{2,1})$ by explicitly constructing a birational contraction of $\overline{M}_{2,1}$. Our main result is an extension of this to higher genera:

Theorem 1.1. *There is a birational contraction of $\overline{M}_{3,1}$ contracting the Weierstrass divisor $BN^1_{3,(0,3)}$. Similarly, there is a birational contraction of $\overline{M}_{4,1}$ contracting the pointed Brill-Noether divisor $BN^1_{3,(0,2)}$.*

As a consequence, we identify an extremal ray of the effective cone.

Corollary 1.2. *For $g = 3, 4$, there is an extremal ray of $\overline{NE}^1(\overline{M}_{g,1})$ generated by a pointed Brill-Noether divisor.*

The proof uses variation of GIT. In particular, we consider the following GIT problem: let Y be a surface and fix a linear equivalence class $|D|$ of curves on Y. Now, let

$$X = \{(p, C) \in Y \times |D| \; |p \in C\}$$

be the universal family over this space of curves. In the case where $(Y, |D|)$ is $(\mathbb{P}^2, |\mathcal{O}(4)|)$ or $(\mathbb{P}^1 \times \mathbb{P}^1, |\mathcal{O}(3, 3)|)$, the quotient of $X//Aut(Y)$ is a birational model for $\overline{M}_{3,1}$ or $\overline{M}_{4,1}$, respectively. By varying the choice of linearization, we obtain a birational model in which the specified divisor is contracted.

The outline of the paper is as follows. In section 2 we provide some background on variation of GIT. In section 3, we develop a tool for studying GIT quotients of families of curves on surfaces. In particular, we construct a large class of divisors on these spaces that are invariant under the automorphism group of the surface, called Hessians. In sections 4 and 5 we then examine separately curves on $\mathbb{P}^2$ and on $\mathbb{P}^1 \times \mathbb{P}^1$, yielding our result in the cases of $g = 3$ and 4.

We plan on discussing similar results for genus 5 and 6 in a later paper.

Acknowledgements This work was prepared as part of my doctoral dissertation under the direction of Sean Keel. I would like to thank him for his abundance of help and suggestions. I would also like to thank Brendan Hassett for his ideas.

2. Variation of GIT

The birational contractions that we construct arise naturally as GIT quotients. This section contains a brief summary of results of Dolgachev-Hu [DH98] and Thaddeus [Tha96] on variation of GIT.

Given a group G acting on a variety X, the GIT quotient $X//G$ is not unique – it depends on the choice of a G-ample line bundle. In particular, if $\mathcal{L} \in Pic^G(X)$, we have

$$X//_{\mathcal{L}}G = Proj \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n})^G.$$

Following Dolgachev and Hu, we will call the set of all G-ample line bundles the ***G*-ample cone**. A study of how the quotient varies with the choice of the G-ample line bundle was carried out independently by Dolgachev-Hu [DH98] and Thaddeus [Tha96]. The following theorem is a summary of some of the results of those papers:

Theorem 2.1. [DH98] [Tha96] *The G-ample cone is divided into a finite number of convex cones, called* ***chambers****. Two line bundles $\mathcal{L}$ and $\mathcal{L}'$ lie in the same chamber if $X^s(\mathcal{L}) = X^{ss}(\mathcal{L}) = X^{ss}(\mathcal{L}') = X^s(\mathcal{L}')$. The chambers are bounded by a finite number of* ***walls****. A line bundle $\mathcal{L}$ lies on a wall if $X^{ss}(\mathcal{L}) \neq X^s(\mathcal{L})$. If $\mathcal{L}$ lies on a wall and $\mathcal{L}'$ lies is an adjacent chamber, then there is a morphism $X//_{\mathcal{L}'}G \to X//_{\mathcal{L}}G$. This map is an isomorphism over the stable locus.*

Both Thaddeus and Dolgachev-Hu examine the maps between quotients at a wall in the G-ample cone. Specifically, let $\mathcal{L}_+$, $\mathcal{L}_-$ be G-ample line bundles in adjacent chambers of the G-ample cone, and define $\mathcal{L}(t) = \mathcal{L}_+^t \otimes \mathcal{L}_-^{1-t}$. Suppose that the line between them crosses a wall precisely at $\mathcal{L}(t_0)$. Following Thaddeus, define

$$X^{\pm} = X^{ss}(\mathcal{L}_{t_0}) \backslash X^{ss}(\mathcal{L}_{\mp})$$

$$X^0 = X^{ss}(\mathcal{L}_{t_0}) \backslash (X^{ss}(\mathcal{L}_{+}) \cup X^{ss}(\mathcal{L}_{-}))$$

Theorem 2.2. [Tha96] *Let $x \in X^0$ be a smooth point of X. Suppose that $G \cdot x$ is closed in $X^{ss}(\mathcal{L}_{t_0})$ and that $G_x \cong \mathbb{C}^*$. Then the natural map $X//_{\mathcal{L}_{\pm}}G \rightarrow X//_{\mathcal{L}_{t_0}}G$ is an isomorphism outside of $X^{\pm}//_{\mathcal{L}_{\pm}}G$. Over a neighborhood of x in $X^0//_{\mathcal{L}_{t_0}}G$, $X^{\pm}//_{\mathcal{L}_{\pm}}G$ are fibrations whose fibers are weighted projective spaces.*

In order to determine whether a point is (semi)stable, we will make frequent use of Mumford's numerical criterion. Given a G-ample line bundle $\mathcal{L}$ and a one-parameter subgroup $\lambda : \mathbb{C}^* \rightarrow G$, it is standard to choose coordinates so that λ acts diagonally on $H^0(X, \mathcal{L})^*$. In other words, it is given by $diag(t^{a_1}, t^{a_2}, \ldots, t^{a_n})$. We will refer to the a_i's as the weights of the $\mathbb{C}^*$ action. For a point $x \in X$, Mumford defines

$$\mu_{\lambda}(x) = min(a_i | x_i \neq 0).$$

Then x is stable (semistable) if and only if $\mu_{\lambda}(x) < 0$ (resp. $\mu_{\lambda}(x) \leq 0$) for every nontrivial 1-parameter subgroup λ of G (see Theorem 2.1 in [MFK94]).

3. Hessians

Here we set up the GIT problem that appears in sections 4 and 5. We also identify a collection of G-invariant divisors that will be useful for analyzing this problem.

Let Y be a smooth projective surface over $\mathbb{C}$, $\mathcal{L}'$ an effective line bundle on Y, and $Z = \mathbb{P}H^0(Y, \mathcal{L}')$. Let

$$X = \{(p, C) \in Y \times Z | p \in C\}.$$

We denote the various maps as in the following diagram:

$$\begin{array}{ccccc} X & \xrightarrow{i} & Y \times Z & \xrightarrow{\pi_1} & Y \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \pi_2} & & \\ Z & \xrightarrow{id} & Z & & \end{array}$$

If $\mathcal{L}'$ is base-point free, then X is a projective space bundle over Y, so it is smooth and $PicX \cong PicY \times \mathbb{Z}$. We will later study the GIT quotients of X by the natural action of $Aut(Y)$.

If C is a curve on Y and $\mathcal{L}$ is another line bundle on Y, then for every point $p \in C$ there are $n + 1 = h^0(C, \mathcal{L}|_C)$ different orders of vanishing of sections $s \in H^0(C, \mathcal{L}|_C)$.

Definition 2. *When written in increasing order,*

$$a_0^{\mathcal{L}}(p) < \cdots < a_n^{\mathcal{L}}(p)$$

the orders of vanishing are called the ***vanishing sequence*** *of $\mathcal{L}$ at p. The* ***weight*** *of $\mathcal{L}$ at p is defined to be $w^{\mathcal{L}}(p) = \sum_{i=0}^{n}(a_i^{\mathcal{L}}(p) - i)$. A point is said to be an* ***$\mathcal{L}$-flex*** *if the weight of $\mathcal{L}$ at the point is nonzero.*

In other words, p is an $\mathcal{L}$-flex if the vanishing sequence of $\mathcal{L}$ at p is anything other than $0 < 1 < \cdots < n$.

Definition 3. *The* ***divisor of $\mathcal{L}$-flexes*** *is $\sum_{p\in C} w^{\mathcal{L}}(p)p$. It corresponds to a section $W_{\mathcal{L}}$ of a certain line bundle called the* ***Wronskian*** *of $\mathcal{L}$. We say that a curve H on Y is an* ***$\mathcal{L}$-Hessian*** *if the restriction of H to C is precisely the divisor of $\mathcal{L}$-flexes.*

Returning to our family of curves $f : X \to Z$ above, suppose that $\mathcal{L}$ is a line bundle on Y such that the pushforward $f_*(\pi_1 \circ i)^*\mathcal{L}$ is locally free of rank $n + 1$. We define a relative $\mathcal{L}$-Hessian to be a divisor $H \subseteq X$ whose restriction to each fiber is the divisor of $f_*(\pi_1 \circ i)^*\mathcal{L}$-flexes. Relative $\mathcal{L}$-Hessians were studied by Cukierman [Cuk97], who shows:

Proposition 3.1. [Cuk97] *The class of the relative $\mathcal{L}$-Hessian is*

$$(n+1)c_1(\pi_1 \circ i)^*\mathcal{L} + \binom{n+1}{2} c_1\Omega^1_{X/Z} - c_1 f^* f_*(\pi_1 \circ i)^*\mathcal{L}.$$

In our particular case, we can determine this class more explicitly.

Corollary 3.2. *For X, Y, and Z as above, the class of the relative $\mathcal{L}$-Hessian is*

$$(n+1)c_1(\pi_1 \circ i)^*\mathcal{L} + \binom{n+1}{2}(c_1\pi_1^*\Omega^1_Y|_X + c_1(\pi_1 \circ i)^*\mathcal{L}' + c_1 f^*\mathcal{O}_Z(1))$$
$$-h^0(Y, \mathcal{L} \otimes \mathcal{L}'^*)(c_1 f^*\mathcal{O}_Z(1)).$$

Proof. We follow the proof in [Cuk97]. If I is the ideal sheaf of X in $Z \times Y$, then we have the exact sequence

$$0 \to I/I^2 \to \pi_1^*\Omega^1_Y|_X \to \Omega^1_{X/Z} \to 0$$

so we have

$$c_1\Omega^1_{X/Z} = c_1\pi_1^*\Omega^1_Y|_X - c_1 I/I^2.$$

Also, X is the scheme of zeros of a section of the line bundle $E = (\pi_1 \circ i)^*\mathcal{L}' \otimes f^*\mathcal{O}_Z(1)$ on $Y \times Z$. Note that $I/I^2 \cong E^* \otimes \mathcal{O}_X = E^*|_X$. It follows that

$$c_1\Omega^1_{X/Z} = c_1(\pi_1 \circ i)^*\Omega^1_Y|_X + c_1 E$$
$$= c_1(\pi_1 \circ i)^*\Omega^1_Y|_X + c_1(\pi_1 \circ i)^*\mathcal{L}' + c_1 f^*\mathcal{O}_Z(1).$$

Now, consider the exact sequence on $Y \times Z$

$$0 \to \pi_1^* L \otimes E^* \to \pi_1^* L \to \pi_1^* L|_X \to 0$$

From the projection formula, we see that

$$\pi_{2*}(\pi_1^* \mathcal{L} \otimes E^*) = H^0(Y, \mathcal{L} \otimes \mathcal{L}'^*) \otimes \mathcal{O}_Z(-1)$$

and $R^1\pi_{2*}(\pi_1^* L \otimes E^*) = 0$. This gives us the exact sequence on Z

$$0 \to \pi_{2*}(\pi_1^* L \otimes E^*) \to \pi_{2*}\pi_1^* L \to \pi_{2*}(\pi_1^* L|_X) \to 0$$

Since the middle term is a trivial bundle, the result follows from Proposition 3.1. □

For the remainder of this section, we identify specific examples that will appear in the arguments to follow.

In section 4 we consider the case that $Y = \mathbb{P}^2$ and $\mathcal{L}' = \mathcal{O}_Y(d)$ for some $d \geq 3$. By the above, we see that for every m and d, a relative $\mathcal{O}_Y(m)$-Hessian H_m exists. Since $c_1 \pi_1^* \Omega_Y^1|_X = \mathcal{O}_X(-3, 0)$, if $m < d$, H_m is cut out by a G-invariant section W_m of

$$\mathcal{O}_X((n+1)m + \binom{n+1}{2}(d-3), \binom{n+1}{2}),$$

where $n + 1 = h^0(Y, \mathcal{L}) = \binom{m+2}{2}$.

In particular, H_1 is cut out by a section $W_1 \in H^0(\mathcal{O}_X(3(d-2), 3))$. W_1 vanishes at (p, C) if C is smooth at p and the tangent line to C at p intersects C with multiplicity at least 3, or if p is a singular point of C. Similarly, H_2 is defined by a section of $W_2 \in H^0(\mathcal{O}_X(15d - 33, 15))$. W_2 vanishes at (p, C) if C is smooth at p and the osculating conic to C at p intersects C with multiplicity at least 6, or if p is a singular point of C.

It is known that $H_2 = H_1 \cup H_2'$ is reducible (see Proposition 6.6 in [CF91]). Indeed, if a line meets C with multiplicity 3 at p, then the double line meets C with multiplicity 6 at p. The points of $H_2' \cap C$ are classically known as the **sextatic points** of C, and H_2' is cut out by a G-invariant section W_2' of $\mathcal{O}_X(12(d - \frac{9}{4}), 12)$. A simple calculation shows that $H_2' \cap C$ also contains those points of C where $w^{\mathcal{O}_C(1)}(p) > 1$. These include singular points and points where the tangent line to C is a **hyperflex** (a line that intersects C at p with multiplicity ≥ 4).

Similarly, in section 5 we consider the case that $Y = \mathbb{P}^1 \times \mathbb{P}^1$, and $\mathcal{L}' = \mathcal{O}_Y(d, d)$. Note that, for every (m_1, m_2, d) with $m_i < d$, a relative $\mathcal{O}_Y(m_1, m_2)$-Hessian H'_{m_1,m_2} exists. In this case, our formulas show that the rank of $f_*(\pi_1 \circ i)^* \mathcal{O}_Y(m_1, m_2)$ is

$$n + 1 = h^0(\mathcal{O}_Y(m_1, m_2)) = (m_1 + 1)(m_2 + 1).$$

Also, since $c_1\pi_1^*\Omega^1_Y|_X = \mathcal{O}_X(-2,-2,0)$, we see that H'_{m_1,m_2} is cut out by a section $W'_{m_1,m_2} \in H^0(\mathcal{O}_X(a_1,a_2,b))$ for

$$a_i = (n+1)m_i + \binom{n+1}{2}(d-2)$$
$$b = \binom{n+1}{2}.$$

Since $\mathbb{P}^1 \times \mathbb{P}^1$ has a natural involution, we know that W'_{m_1,m_2} cannot be G-invariant if $m_1 \neq m_2$. Notice, however, that $W'_{m_1,m_2} \otimes W'_{m_2,m_1}$ is a G-invariant section of $\mathcal{O}_X(a,a,b)$ for

$$n+1 = (m_1+1)(m_2+1)$$
$$a = (n+1)(m_1+m_2) + 2\binom{n+1}{2}(d-2)$$
$$b = 2\binom{n+1}{2}.$$

We will use W_{m_1,m_2} to denote the G-invariant section described here, and H_{m_1,m_2} to denote its zero locus.

In particular, $W_{0,1} \in H^0(\mathcal{O}_X(2(d-1),2(d-1),2))$. It vanishes at a point (p,C) if C intersects one of the two lines through p with multiplicity at least 2 (or, equivalently, if the osculating $(1,1)$ curve is a pair of lines). Similarly, $W_{1,1} \in H^0(\mathcal{O}_X(2(3d-4),2(3d-4),6))$. It vanishes at a point (p,C) if there is a curve of bidegree $(1,1)$ that intersects C with multiplicity 4 or more at p.

4. Contraction of $\overline{M}_{3,1}$

In this section, we prove our main result in the genus 3 case:

Theorem 4.1. *There is a birational contraction of $\overline{M}_{3,1}$ contracting the Weierstrass divisor $BN^1_{3,(0,3)}$.*

In order to construct a birational model for $\overline{M}_{3,1}$, we consider GIT quotients of the universal family over the space of plane quartics. The image of the Weierstrass divisor in this model is precisely the Hessian H_1, and we exhibit a GIT quotient in which this locus is contracted. For most of this section we will consider, more generally, plane curves of any degree $d \geq 3$.

Specifically, following the set-up of the previous section, we let

$$X = \{(p,C) \in \mathbb{P}^2 \times |\mathcal{O}(d)| \; |p \in C\}.$$

Then $\pi_2 : X \to |\mathcal{O}(d)|$ is the family of all plane curves of degree d. Our goal is to study the GIT quotients of X by the action of $G = PSL(3,\mathbb{C})$. By the above, we know that $PicX \cong \mathbb{Z} \times \mathbb{Z}$, so the quotient $X//_{\mathcal{L}}G$ depends on a single parameter t which we call the **slope** of $\mathcal{L}$.

Definition 4. *We say a line bundle $\mathcal{L}$ has* ***slope*** *t if $\mathcal{L} = \pi_1^*\mathcal{O}(a) \otimes \pi_2^*\mathcal{O}(b)$ with $t = \frac{a}{b}$. We write $X^s(t)$ and $X^{ss}(t)$ for the sets of stable and semistable points, and $X//_tG$ for the corresponding GIT quotient.*

Here we describe the numerical criterion for points in X. Let $p = (x_0, x_1, x_2)$ and

$$C = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k.$$

Then a basis for $H^0(\mathcal{O}_X(a,b))$ consists of monomials of the form

$$\prod_{\alpha=1}^{a} x_{l_\alpha} \prod_{\beta=1}^{b} a_{i_\beta, j_\beta, k_\beta}.$$

The one-parameter subgroup with weights (r_0, r_1, r_2) acts on the monomial above with weight

$$\sum_{\alpha=1}^{a} r_{l_\alpha} - \sum_{\beta=1}^{b} (i_\beta r_0 + j_\beta r_1 + k_\beta r_2).$$

In our case, we will only be interested in maximizing or minimizing this weight, so it suffices to consider monomials of the form $x_l^a a_{i,j,k}^b$. In this case, the one-parameter subgroup acts with weight $ar_l - b(ir_0 + jr_1 + kr_2)$, which is proportional to

$$\mu_\lambda(x_l, a_{i,j,k}) := tr_l - (ir_0 + jr_1 + kr_2).$$

The G-ample cone of X has two edges, one of which occurs when $t = 0$. In the case where $d = 4$, we obtain the well-known moduli space of plane quartics. Descriptions of $X^s(0)$ and $X^{ss}(0)$ appear in [MFK94], and the quotient $X//_0G$ plays an important role in the birational geometry of $\overline{M}_3$. For example, Hyeon and Lee show that this quotient is a log canonical model for $\overline{M}_3$ [HL10], and the space also appears in work on moduli of K3 surfaces [Art09] and cubic threefolds [CML09].

We will see that, when t is large, stability conditions reflect the inflectionary behavior of linear series at the marked point. Thus, as t increases, the curve is allowed to have more complicated singularities, but vanishing sequences at the marked point become more well-behaved.

Our first result is to identify the other edge of the G-ample cone. It is determined by the Wronksian W_1.

Proposition 4.2. *An edge of the G-ample cone occurs at $t = d - 2$.*

Proof. It suffices to show that $X^{ss}(d-2) \neq X^s(d-2) = \emptyset$. It is clear that $X^{ss}(d-2) \neq \emptyset$, since W_1 is a G-invariant section of $\mathcal{O}_X(3(d-2), 3)$.

To show that $X^s(d-2) = \emptyset$, we invoke the numerical criterion. Let $(p, C) \in X$. By change of coordinates, we may assume that $p = (0, 0, 1)$

and the tangent line to C at p is $x_0 = 0$. So in the coordinates described above, we have $a_{0,0,d} = a_{0,1,d-1} = 0$.

Now consider the 1-parameter subgroup with weights $(-1, 0, 1)$. We have

$$\mu_\lambda(x_2, a_{i,j,k}) = d - 2 + i - k$$

which is negative whenever $i - k - 2 < -d = -i - j - k$, or $2i + j < 2$. This only occurs when both $i = 0$ and $j < 2$, in other words, when either $a_{0,0,d}$ or $a_{0,1,d-1}$ is nonzero. By assumption, however, this is not the case, so $(p, C) \notin X^s(d - 2)$. Since (p, C) was arbitrary, it follows that $X^s(d - 2) = \emptyset$.

□

Next, we identify the adjacent chamber in the G-ample cone. It lies between the slopes corresponding to W_1 and W_2'. In what follows, we let S denote the set of all pointed curves (p, C) admitting the following description: C consists of a smooth conic together with $d - 2$ copies of the tangent line through a point $q \neq p$ on C. Notice that $S \subset H_2'$.

Proposition 4.3. *For any* $t \in (d - \frac{9}{4}, d - 2)$, $X^s(t) = X^{ss}(t) = X \backslash (H_1 \cup S)$.

Proof. We first show that $X^{ss}(t) \subseteq X \backslash H_1$. Suppose that $(p, C) \in H_1$. As before, by change of coordinates, we may assume that $p = (0, 0, 1)$ and the tangent line to C at p is $x_0 = 0$. Since $(p, C) \in H_1$, either p is a singular point of C or this tangent line intersects C at p with multiplicity at least 3. Thus we have $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ (if p is singular) or $a_{0,2,d-2} = 0$ (if p is a flex).

We first examine the case where p is a flex. In this case, consider the 1-parameter subgroup with weights $(-5, 1, 4)$. Then

$$\mu_\lambda(x_2, a_{i,j,k}) = 4t + 5i - j - 4k > 4d - 9 + 5i - j - 4k = 9i + 3j - 9$$

which is non-negative when $3i + j \geq 3$. Since, by assumption, C has no non-zero terms with both $i = 0$ and $j < 3$, we see that $(p, C) \notin X^{ss}(t)$.

Next we look at the case where p is a singular point. Consider the 1-parameter subgroup with weights $(-1, -1, 2)$. Then we have

$$\mu_\lambda(x_2, a_{i,j,k}) = 2t + i + j - 2k > 2d - \frac{9}{2} + i + j - 2k = 3i + 3j - \frac{9}{2}$$

which is non-negative when $i + j \geq \frac{3}{2}$. By assumption, C has no non-zero terms where one of i, j is 0 and the other is at most 1, so $(p, C) \notin X^{ss}(t)$. It follows that $X^{ss}(t) \subseteq X \backslash H_1$.

Next we show that $X^{ss}(t) \subseteq X \backslash S$. Suppose that $(p, C) \in S$. Without loss of generality, we may assume that C is of the form

$$C = x_0^{d-2}(a_{d,0,0}x_0^2 + a_{d-1,1,0}x_0x_1 + a_{d-2,2,0}x_1^2 + a_{d-1,0,1}x_0x_2).$$

Now, consider the 1-parameter subgroup with weights $(-1, 0, 1)$. Then

$$\mu_\lambda(x_l, a_{i,j,k}) \geq -t + i - k > 2 - d + i - k$$

which is non-negative when $i-k \geq d-2$. It follows that $(p, C) \notin X^{ss}(t)$.

Now we show that $X \backslash (H_1 \cup S) \subseteq X^s(t)$. Suppose that $(p, C) \notin X^s(t)$. Then there is a nontrivial 1-parameter subgroup that acts on (p, C) with non-negative weight. By change of basis, we may assume that this subgroup acts with weights (r_0, r_1, r_2), with $r_0 \leq r_1 \leq r_2$. Since this is a nontrivial subgroup of $PSL(3, \mathbb{C})$, we know that $r_0 < 0 < r_2$ and $r_0 + r_1 + r_2 = 0$. We then have

$$\mu_\lambda(x_l, a_{i,j,k}) = tr_l - (r_0 i + r_1 j + r_2 k) \geq 0$$

We divide this into three cases, depending on p.

Case 1 – $p = (0, 0, 1)$: In this case, $r_l = r_2$. If $r_1 \geq 0$, then $tr_2 < (d-2)r_2 \leq 2r_1 + (d-2)r_2$. On the other hand, if $r_1 < 0$¡ then $tr_2 < (d-2)r_2 < r_0 + (d-1)r_2$. Since the subgroup acts with non-negative weight, it follows that $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ or $a_{0,2,d-2} = 0$. Hence, $(p, C) \in H_1$.

Case 2 – p lies on the line $x_0 = 0$, but not on the line $x_1 = 0$: In this case, $r_l = r_1$. If $r_1 > 0$, then since $r_1 \leq r_2$, we have $tr_1 < dr_1 \leq r_1 j + r_2(d-j)$, so we see that $a_{0,0,d} = a_{0,1,d-1} = \cdots = a_{0,d,0} = 0$. This means that p lies on a linear component of C, and therefore $(p, C) \in H_1$.

On the other hand, if $r_1 \leq 0$, then since $r_2 \geq -2r_1$, we see that $tr_1 \leq (d-3)r_1 \leq (d-1)r_1 + r_2 \leq r_1 j + (d-j)r_2 + r_2$ for $j \leq d-1$. Note furthermore that if $r_1 < 0$, then the first of these inequalities is strict, whereas if $r_1 = 0$, the second inequality is strict. It follows that $a_{0,0,d} = a_{0,1,d-1} = \cdots = a_{0,d-1,1} = 0$. This means that either p lies on a linear component of C or the only point of C lying on the line $x_0 = 0$ also lies on the line $x_1 = 0$. Again, we see that $(p, C) \in H_1$.

Case 3 – p does not lie on the line $x_0 = 0$: In this case, $r_l = r_0$. Since $r_0 < 0$ and $r_0 \leq r_1 \leq r_2$, we see that $tr_0 < (d-3)r_0 = (d-2)r_0 + r_1 + r_2 < r_0 i + r_1 j + r_2 k$ for $i \leq d-2, k \neq 0$. Now, if $r_0 \geq 4r_1$, then we have $tr_0 < (d - \frac{9}{4})r_0 = (d - \frac{5}{4})r_0 + r_1 + r_2 \leq (d-1)r_0 + r_2$. It follows that C is of the form

$$C = \sum_{i+j=d} a_{i,j,0} x_0^i x_1^j.$$

In other words, C is a union of d lines. In this case, the tangent line to every point of C is a component of C itself, so $(p, C) \in H_1$.

On the other hand, if $r_0 < 4r_1$, then $tr_0 < (d-\frac{9}{4})r_0 = (d-3)r_0 + \frac{3}{4}r_0 < (d-3)r_0 + 3r_1$. It follows that C is of the form

$$C = x_0^{d-2}(a_{d,0,0}x_0^2 + a_{d-1,1,0}x_0 x_1 + a_{d-2,2,0}x_1^2 + a_{d-1,0,1}x_0 x_2)$$

hence $C \in S$.

□

We now consider the wall in the G-ample cone determined by W_2'.

Proposition 4.4. *A wall of the G-ample cone occurs at $t = d - \frac{9}{4}$. More specifically, $X^{ss}(t) = X \backslash ((H_1 \cap H_2') \cup S)$, and $X^s(t) \subseteq X \backslash (H_1 \cup S)$.*

Proof. First, notice that if $(p, C) \notin H_2'$, then $(p, C) \in X^{ss}(t)$, since W_2' is a G-invariant section of $\mathcal{O}_X(12(d - \frac{9}{4}), 12)$ that does not vanish at (p, C). Moreover, by general variation of GIT we know that, when passing from a chamber to a wall, we have

$$X^{ss}(t + \epsilon) \subseteq X^{ss}(t)$$

$$X^s(t) \subseteq X^s(t + \epsilon)$$

Thus, $X^s(t) \subseteq X \backslash (H_1 \cup S)$ and $X \backslash ((H_1 \cap H_2') \cup S) \subseteq X^{ss}(t)$.

Now, suppose that $(p, C) \in S$. Using the same argument as above with the same 1-parameter subgroup, we see that $(p, C) \notin X^{ss}(t)$.

Next, suppose that $(p, C) \in H_1$. If p is a singular point of C, then we see that $(p, C) \notin X^{ss}(t)$ by the same argument as before, using the subgroup with weights $(-1, -1, 2)$.

The only other possibility is that p is a flex. In this case, we again consider the 1-parameter subgroup with weights $(-5, 1, 4)$. As before, we have

$$\mu_\lambda(x_2, a_{i,j,k}) = 4d - 9 + 5i - j - 4k = 9i + 3j - 9$$

which is non-negative when $3i + j \geq 3$. As before, we see that $(p, C) \notin X^s(t)$.

Notice furthermore that if $(p, C) \in H_1 \cap H_2'$, then either $a_{0,3,d-3} = 0$ or $a_{1,0,d-1} = 0$. Now consider the 1-parameter subgroup with weights $(-5 - \epsilon, 1 + \epsilon, 4)$. For $\epsilon > 0$, we see that any curve with $a_{0,3,d-3} = 0$ is unstable. Conversely, if $\epsilon < 0$, we see that any curve with $a_{1,0,d-1} = 0$ is unstable. From our observations above, we may therefore conclude that $X^{ss}(t) \subseteq X \backslash ((H_1 \cap H_2') \cup S)$.

□

We are left to consider the behavior of our quotient at the wall crossing defined by $t_0 = d - \frac{9}{4}$. As in Theorem 2.2, we let

$$X^{\pm} = X^{ss}(t_0) \backslash X^{ss}(t_0 \mp \epsilon)$$

$$X^0 = X^{ss}(t_0) \backslash (X^{ss}(t_0 + \epsilon) \cup X^{ss}(t_0 - \epsilon))$$

Our first task is to determine X^- and X^0 in this situation.

Proposition 4.5. *With the set-up above, $X^- = H_1 \backslash H_2'$. X^0 is the set of all pointed curves (p, C) consisting of a cuspidal cubic plus $d - 3$ copies of the projectivized tangent cone at the cusp. The point p is the unique smooth flex point of the cuspidal cubic.*

Proof. We have already seen that $X^{ss}(t_0) = X\backslash((H_1 \cap H_2') \cup S)$ and $X^{ss}(t_0+\epsilon) = X\backslash(H_1 \cup S)$. Thus, $X^- = H_1\backslash H_2'$.

To prove the statement about X^0, let $(p, C) \in X^0$. Notice that, since $X^0 \subseteq X^-$, p is a smooth point of C and the tangent line to C at p intersects C with multiplicity exactly 3. Since $(p, C) \notin X^{ss}(t_0 - \epsilon)$, there must be a nontrivial 1-parameter subgroup that acts on (p, C) with strictly positive weight. Again we assume that this subgroup acts with weights (r_0, r_1, r_2), with $r_0 \leq r_1 \leq r_2$. As before, we know that $r_0 < 0 < r_2$ and $r_0 + r_1 + r_2 = 0$. Again we have

$$\mu_\lambda(x_l, a_{i,j,k}) = tr_l - (r_0 i + r_1 j + r_2 k) > 0$$

We divide this into three cases, depending on p.

Case 1 – $p = (0, 0, 1)$: In this case, $r_l = r_2$. Now, if $tr_2 \geq r_0 + (d-1)r_2$, then $(d - \frac{9}{4})r_2 > r_0 + (d-1)r_2$, so $r_1 > \frac{1}{4}r_2$. This means that $tr_2 < (d - \frac{9}{4})r_2 < 3r_1 + (d-3)r_2$. It follows that $a_{0,0,d} = a_{0,1,d-1} = 0$, and either $a_{1,0,d-1} = 0$ or $a_{0,2,d-2} = a_{0,3,d-3} = 0$. But we know that p is a smooth point of C and the tangent line to C at p intersects C with multiplicity exactly 3, so neither of these is a possibility.

Case 2 – p lies on the line $x_0 = 0$, but not on the line $x_1 = 0$: Using the same argument as before, we see that p lies on a linear component of C, which is impossible.

Case 3 – p does not lie on the line $x_0 = 0$: In this case, $r_l = r_0$. Again, since $r_0 < 0$ and $r_1 < r_0 < r_2$, we see that $tr_0 < (d-3)r_0 = (d-2)r_0 + r_1 + r_2 < r_0 i + r_1 j + r_2 k$ for $i \leq d-2, k \neq 0$. Notice that, if $tr_0 < (d-1)r_0 + r_2$, then as before we see that C is the union of d lines, which is impossible.

We therefore see that $(d - \frac{12}{5})r_0 > tr_0 \geq (d-1)r_0 + r_2$. But then $\frac{7}{5}r_0 < -r_2 = r_0 + r_1$, so $r_0 < \frac{5}{2}r_1$. It follows that $tr_0 < (d - \frac{12}{5})r_0 < (d-4)r_0 + 4r_1 \leq r_0 i + r_1 j$ for $j \geq 4$.

We see that C is of the form

$$C = x_0^{d-3}(a_{d,0,0}x_0^3 + a_{d-1,1,0}x_0^2 x_1 + a_{d-2,2,0}x_0 x_1^2 + a_{d-3,3,0}x_1^3 + a_{d-1,0,1}x_0^2 x_2).$$

Thus, C consists of a cuspidal cubic together with $d-3$ copies of the projectivized tangent cone to the cusp. The point p is the unique flex point of the cuspidal cubic.

It is clear that this $(p, C) \in X^-$, since the tangent line to C at p intersects C with multiplicity exactly 3. To see that $(p, C) \notin X^{ss}(t_0 - \epsilon)$, consider again the 1-parameter subgroup with weights $(5, -1, -4)$. The characterization of X^0 above then follows from the fact that all cuspidal plane cubics are projectively equivalent.

□

Corollary 4.6. *The map $X//_{t_0-\epsilon}G \to X//_{t_0}G$ contracts the locus $H_1\backslash H_2'$ to a point. Outside of this locus, the map is an isomorphism.*

Proof. Let $(p, C) \in X^0$. Since all cuspidal plane cubics are projectively equivalent, $G \cdot (p, C) = X^0$, so $G \cdot (p, C)$ is closed in $X^{ss}(t_0)$ and $X^0//G$ is a point. An automorphism of $\mathbb{P}^1$ extends to (p, C) if and only if it fixes the point p and the cusp, and thus the stabilizer of (p, C) is isomorphic to $\mathbb{C}^*$. The conclusion follows from Theorem 2.2. □

We are particularly interested in the case where $d = 4$, because in this case $X//_{t_0-\epsilon}G$ is a birational model for $\overline{M}_{3,1}$. In particular, we have the following:

Proposition 4.7. *There is a birational contraction* $\beta : \overline{M}_{3,1} \dashrightarrow X//_{t_0-\epsilon}G$.

Proof. It suffices to exhibit a morphism $\beta^{-1} : V \to \overline{M}_{3,1}$, where $V \subseteq X//_{t_0-\epsilon}G$ is open with complement of codimension ≥ 2 and β^{-1} is an isomorphism onto its image. To see this, let $U \subseteq X^{ss}(t_0 - \epsilon)$ be the set of all moduli stable pointed curves $(p, C) \in X^{ss}(t_0 - \epsilon)$. Notice that U is invariant under the action of the group and its complement is strictly contained in the discriminant locus Δ, which is an irreducible G-invariant hypersurface in X. Note furthermore that there are stable points contained in both $X \backslash \Delta$ and $\Delta \cap U$. Thus, the containments $(X \backslash U)//_{t_0-\epsilon}G \subset \Delta//_{t_0-\epsilon}G$ and $\Delta//_{t_0-\epsilon}G \subset X//_{t_0-\epsilon}G$ are strict. It follows that the complement of $U//G$ in the quotient has codimension ≥ 2.

By the universal property of the moduli space, since U is a family of moduli stable curves, it admits a unique map $U \to \overline{M}_{3,1}$. Since U is contained in the semistable locus and this map is G-equivariant, it factors uniquely through a map $U//_{t_0-\epsilon}G \to \overline{M}_{3,1}$. Since every degree 4 plane curve is canonical, two such curves are isomorphic if and only if they differ by an automorphism of $\mathbb{P}^2$. It follows that this map is an isomorphism onto its image. □

Theorem 4.8. *There is a birational contraction of $\overline{M}_{3,1}$ contracting the Weierstrass divisor $BN^1_{3,(0,3)}$. Furthermore, the divisors $BN^1_{3,(0,3)}$, BN^1_2, Δ_1 and Δ_2 span a simplicial face of $\overline{NE}^1(\overline{M}_{3,1})$.*

Proof. The composition $\overline{M}_{3,1} \dashrightarrow X//_{t_0-\epsilon}G \to X//_{t_0}G$ is a birational contraction. By the above, the Weierstrass divisor is contracted by this map. It therefore suffices to show that the isomorphism β^{-1} constructed in the preceding theorem does not contain in its image the generic point of BN^1_2 or Δ_i for $i \geq 1$. For BN^1_2 this is automatic, since every smooth curve in X is canonically embedded and hence non-hyperelliptic. For Δ_i this follows directly from the fact that $\Delta \cap U$ is an irreducible divisor in U whose generic point is an irreducible nodal curve.

□

5. Contraction of $\overline{M}_{4,1}$

We now turn to the case of genus 4 curves. Our main result will be the following:

Theorem 5.1. *There is a birational contraction of $\overline{M}_{4,1}$ contracting the pointed Brill-Noether divisor $BN^1_{3,(0,2)}$.*

In a similar way to the previous section, we will construct a birational model for $\overline{M}_{4,1}$ by considering GIT quotients of the universal family over the space of curves in $\mathbb{P}^1 \times \mathbb{P}^1$. Here, the Hessian $H_{0,1}$ is again the image of a pointed Brill-Noether divisor. As above, our goal is to find a GIT quotient in which this locus is contracted. Let $Y = \mathbb{P}^1 \times \mathbb{P}^1$ and

$$X = \{(p, C) \in Y \times |\mathcal{O}(d,d)| \, | p \in C\}.$$

Then $\pi_2 : X \to |\mathcal{O}(d,d)|$ is the family of all curves of bidegree (d,d). Our goal, as before, is to study the GIT quotients of X by the action of $G = PSO(4, \mathbb{C})$. By the above, we know that $PicX \cong \mathbb{Z}^3$, but we are only interested in those line bundles of the form $\mathcal{O}_X(a,a,b)$. We can therefore define the slope of a line bundle $\mathcal{L} \in PicX$ as above.

Definition 5. *We say a line bundle $\mathcal{L}$ has* ***slope*** *t if $\mathcal{L} = \pi_1^*\mathcal{O}(a,a) \otimes \pi_2^*\mathcal{O}(b)$ with $t = \frac{a}{b}$. We write $X^s(t)$ and $X^{ss}(t)$ for the sets of stable and semistable points, and $X//_tG$ for the corresponding GIT quotient.*

Here we describe the numerical criterion for points in X. Let $p = (x_0, x_1 : y_0, y_1)$ and

$$C = \sum_{0 \le i,j \le d} a_{i,j} x_0^i x_1^{d-i} y_0^j y_1^{d-j}.$$

Then a basis for $H^0(\mathcal{O}_X(a,a,b))$ consists of monomials of the form

$$\prod_{\alpha_0=1}^{a} x_{l_{\alpha_0}} y_{m_{\alpha_1}} \prod_{\beta=1}^{b} a_{i_\beta, j_\beta}.$$

The one-parameter subgroup with weights $(-r_0, r_0, -r_1, r_1)$ acts on the monomial above with weight

$$\sum_{\beta=1}^{b} (r_0(i_\beta - (d - i_\beta)) + r_1(j_\beta - (d - j_\beta))) - \sum_{\alpha_0=1}^{a} ((-1)^{l_{\alpha_0}} r_0 + (-1)^{m_{\alpha_1}} r_1).$$

In our case, we will only be interested in maximizing or minimizing this weight, so it suffices to consider monomials of the form $x_l^a y_m^a a_{i,j}^b$. In this case, the one-parameter subgroup acts with weight $b(r_0(2i-d) + r_1(2j-d)) - a((-1)^l r_0 + (-1)^m r_1)$, which is proportional to

$$\mu_\lambda(x_l, y_m, a_{i,j}) := r_0(2i-d) + r_1(2j-d) - t((-1)^l r_0 + (-1)^m r_1).$$

As in the previous section, when $t = 0$, we obtain a moduli space of curves of bidegree (d, d). In particular, the case $d = 3$ is notable for being a birational model for $\overline{M}_4$. We will see that as t increases, stable curves are allowed to have more complicated singularities, but the vanishing sequences of linear series at the marked point become more well-controlled. We begin by identifying an edge of the G-ample cone corresponding to the Wronskian $W_{0,1}$.

Proposition 5.2. *An edge of the G-ample cone occurs at $t = d - 1$.*

Proof. It suffices to show that $X^{ss}(d-1) \neq X^s(d-1) = \emptyset$. It is clear that $X^{ss}(d-1) \neq \emptyset$, since $W_{0,1}$ is a G-invariant section of $\mathcal{O}_X(2(d-1), 2(d-1), 2)$.

To show that $X^s(d-1) = \emptyset$, we invoke the numerical criterion. Let $(p, C) \in X$. By change of coordinates, we may assume that $p = (0, 1 : 0, 1)$. So, in the coordinates described above, we have $a_{0,0} = 0$.

Now consider the 1-parameter subgroup with weights $(-1, 1, -1, 1)$. We have

$$\mu_\lambda(x_1, y_1, a_{i,j}) = 2(d-1) + (2i - d) + (2j - d)$$

which is negative whenever $(2i-d)+(2j-d) < -2(d-1)$, or $i+j < 1$. This only occurs when $i = j = 0$, in other words, when $a_{0,0}$ is nonzero. By assumption, however, this is not the case, so $(p, C) \notin X^s(d-1)$. Since (p, C) was arbitrary, it follows that $X^s(d-1) = \emptyset$. □

As above, we identify the adjacent chamber in the G-ample cone. It lies between the slopes corresponding to the Wronskians $W_{0,1}$ and $W_{1,1}$. In what follows, we let S denote the set of all pointed curves (p, C) admitting the following description: C consists of a smooth curve of bidegree $(1, 1)$ together with $d - 1$ copies of the two lines through a point $q \neq p$ on C. Notice that $S \subset H_{1,1}$.

Proposition 5.3. *For any $t \in (d - \frac{4}{3}, d - 1)$, $X^s(t) = X^{ss}(t) = X \backslash (H_{0,1} \cup S)$.*

Proof. We first show that $X^{ss}(t) \subseteq X \backslash H_{0,1}$. Suppose that $(p, C) \in H_{0,1}$. As before, by change of coordinates, we may assume that $p = (0, 1 : 0, 1)$. Since $(p, C) \in H_{0,1}$, C intersects one of the two lines through p with multiplicity at least 2. Without loss of generality, we may assume this line to be $x_0 = 0$. Thus, if we write C as above, then $a_{0,0} = a_{0,1} = 0$. Now, consider the 1-parameter subgroup with weights $(-2, 2, -1, 1)$. Then

$$\mu_\lambda(x_1, y_1, a_{i,j}) = 3t + 2(2i-d) + (2j-d) > 3d - 4 + 2(2i-d) + (2j-d)$$
$$= 2(2i + j - 2)$$

which is non-negative when $2i+j \geq 2$. Since, by assumption, C has no non-zero terms with both $i = 0$ and $j \leq 1$, we see that $(p, C) \notin X^{ss}(t)$.

Next we show that $X^{ss}(t) \subseteq X \backslash S$. Suppose that $(p, C) \in S$. Without loss of generality, we may assume that C is of the form

$$C = x_0^{d-1} y_0^{d-1} (a_{d,d} x_0 y_0 + a_{d-1,d} x_1 y_0 + a_{d,d-1} x_0 y_1).$$

Now, consider the 1-parameter subgroup with weights $(1, -1, 1, -1)$. Then

$$\mu_\lambda(x_l, y_m, a_{i,j}) \geq -2t - (2i-d) - (2j-d) > -2d+2-(2i-d)-(2j-d)$$
$$= -2((d-i) + (d-j) - 1)$$

which is non-negative when $(d-i) + (d-j) \leq 1$. It follows that $(p, C) \notin X^{ss}(t)$.

Now we show that $X \backslash (H_{0,1} \cup S) \subseteq X^s(t)$. Suppose that $(p, C) \notin X^s(t)$. Then there is a nontrivial 1-parameter subgroup that acts on (p, C) with non-negative weight. By change of basis, we may assume that this subgroup acts with weights $(-r_0, r_0, -r_1, r_1)$, with $0 \leq r_0 \leq r_1$ and $r_1 > 0$. We then have

$$\mu_\lambda(x_l, y_m, a_{i,j}) = r_0(2i-d) + r_1(2j-d) - t((-1)^l r_0 + (-1)^m r_1) \geq 0$$

We divide this into four cases, depending on p.

Case 1 – $p = (0, 1 : 0, 1)$: In this case, $l = m = 1$. We have $t(-r_0 - r_1) > (d-1)(-r_0 - r_1) \geq -(d-2)r_0 - dr_1$. It follows that $a_{0,0} = a_{1,0} = 0$, so $(p, C) \in H_{0,1}$.

Case 2 – p lies on the line $y_0 = 0$, but not the line $x_0 = 0$: In this case, $l = 1$ and $m = 0$. Here, $t(-r_0 + r_1) \geq (d-2)(-r_0 + r_1) \geq -dr_0 + kr_1$ for all $k \leq d-2$. Note further that if $r_0 \neq r_1$, then the first inequality is strict, whereas if $r_0 = r_1$, then the second inequality is strict. We therefore see that $a_{0,k} = 0$ for all $k \leq d-2$. If $a_{0,d} \neq 0$, then every point of C that lies on the line $x_0 = 0$ also lies on the line $y_0 = 0$, a contradiction. We therefore see that $a_{0,d} = 0$ as well, but this means that p lies on a linear component of C, and therefore $(p, C) \in H_{0,1}$.

Case 3 – p lies on the line $x_0 = 0$, but not on the line $y_0 = 0$: In this case, $l = 0$ and $m = 1$. Note that $t(r_0 - r_1) \geq d(r_0 - r_1) \geq dr_0 - kr_1$ for all $k < d$. Again, if $r_0 \neq r_1$, then the first inequality is strict, whereas if $r_0 = r_1$, then the second inequality is strict. It follows that $a_{k,0} = 0$ for all $k < d$, which means that either $y_0 = 0$ is a linear component of C or every point of C lies on the line $y_0 = 0$ also lies on the line $y_0 = 0$. Thus $(p, C) \in H_{0,1}$.

Case 4 – p does not lie on either of the lines $x_0 = 0$ or $y_0 = 0$: In this case, $l = m = 0$. Now note that $t(r_0 + r_1) > (d-2)(r_0 + r_1)$, so $a_{k_0,k_1} = 0$ if k_0 and k_1 are both less than d. Furthermore, since $r_0 \leq r_1$, $(d-2)(r_0 + r_1) \geq dr_0 + (d-4)r_1$, so $a_{d,k} = 0$ for $k \leq d-2$. Now, if $(d - \frac{4}{3})(r_0 + r_1) \leq (d-4)r_0 + dr_1$, then $2r_0 \leq r_1$, so $t(r_0 + r_1) >$

$(d-\frac{4}{3})(r_0+r_1) \geq dr_0 + (d-2)r_1$. It follows that either $a_{d,d-1}=0$, in which case C is a union of $2d$ lines and hence $(p,C) \in H_{0,1}$, or $a_{k,d}=0$ for all $k \leq d-2$, in which case $C \in S$.

□

We now consider the wall in the G-ample cone determined by the Wronskian $W_{1,1}$.

Proposition 5.4. *A wall of the G-ample cone occurs at $t = d-\frac{4}{3}$. More specifically, $X^{ss}(t) = X \backslash ((H_{0,1} \cap H_{1,1}) \cup S)$, and $X^s(t) \subseteq X \backslash (H_{0,1} \cup S)$.*

Proof. First, notice that if $(p,C) \notin H_{1,1}$, then $(p,C) \in X^{ss}(t)$, since $W_{1,1}$ is a G-invariant section of $\mathcal{O}_X(6(d-\frac{4}{3}), 6(d-\frac{4}{3}), 6)$ that does not vanish at (p,C). Moreover, by general variation of GIT we know that, when passing from a chamber to a wall, we have

$$X^{ss}(t+\epsilon) \subseteq X^{ss}(t)$$

$$X^s(t) \subseteq X^s(t+\epsilon)$$

Thus, $X^s(t) \subseteq X \backslash (H_{0,1} \cup S)$ and $X \backslash ((H_{0,1} \cap H_{1,1}) \cup S) = X^{ss}(t)$.

Now, suppose that $(p,C) \in S$. Using the same argument as before with the same 1-parameter subgroup, we see that $(p,C) \notin X^{ss}(t)$.

Next, suppose that $(p,C) \in H_{0,1}$. In this case, we again consider the 1-parameter subgroup with weights $(-2,2,-1,1)$. As before, we have

$$\mu_\lambda(x_1,y_1,a_{i,j}) = 3d-4+2(2i-d)+(2j-d) = 2(2i+j-2)$$

which is non-negative when $2i+j \geq 2$. Since, by assumption, C has no non-zero terms with both $i=0$ and $j \leq 1$, we see that $(p,C) \notin X^s(t)$.

Notice furthermore that if $(p,C) \in H_{0,1} \cap H_{1,1}$, this means that the osculating $(1,1)$ curve to C at p is the pair of lines through that point, and this curve intersects C with multiplicity at least 4. This means that either $a_{0,1}=0$ or $a_{2,0}=0$, which implies that the expression $2i+j-2$ above is zero for at most one term, and strictly positive for all of the others. Now consider the 1-parameter subgroup with weights $(-2-\epsilon, 2+\epsilon, -1, 1)$. For $\epsilon > 0$, we see that any curve with $a_{0,1}=0$ is unstable. Conversely, if $\epsilon < 0$, we see that any curve with $a_{2,0}=0$ is unstable. It follows that $(p,C) \notin X^{ss}(t)$, and thus $X^{ss}(t) = X \backslash ((H_{0,1} \cap H_{1,1}) \cup S)$.

□

Again, we want to use Theorem 2.2 to study the wall crossing at $t_0 = d-\frac{4}{3}$. Again, we let

$$X^{\pm} = X^{ss}(t_0) \backslash X^{ss}(t_0 \mp \epsilon)$$

$$X^0 = X^{ss}(t_0) \backslash (X^{ss}(t_0+\epsilon) \cup X^{ss}(t_0-\epsilon))$$

and determine X^- and X^0.

Proposition 5.5. *With the set-up above, $X^- = H_{0,1}\backslash H_{1,1}$. X^0 is the set of all pointed curves (p, C) admitting the following description: C consists of a smooth curve of bidegree $(1, 2)$ (or $(2, 1)$), together with $d-1$ copies of the tangent line to this curve through a point that has a tangent line, and $d-2$ copies of the other line through this same point. The marked point p is the unique other point on the smooth $(1, 2)$ curve that has a tangent line.*

Proof. We have already seen that $X^{ss}(t_0) = X\backslash((H_{0,1} \cap H_{1,1}) \cup S)$ and $X^{ss}(t_0 + \epsilon) = X\backslash(H_{0,1} \cup S)$. Thus, $X^- = H_{0,1}\backslash H_{1,1}$.

To prove the statement about X^0, let $(p, C) \in X^0$. Notice that, since $X^0 \subseteq X^-$, exactly one of the two lines through p intersects C with multiplicity exactly 2. Since $(p, C) \notin X^{ss}(t_0 - \epsilon)$, there must be a nontrivial 1-parameter subgroup that acts on (p, C) with strictly positive weight. Again we assume that this subgroup acts with weights $(-r_0, r_0, -r_1, r_1)$, with $0 \leq r_0 \leq r_1$ and $r_1 > 0$. Again we have

$$\mu_\lambda(x_l, y_m, a_{i,j}) = r_0(2i - d) + r_1(2j - d) - t((-1)^l r_0 + (-1)^m r_1) > 0$$

We divide this into four cases, depending on p.

Case 1 – $p = (0, 1 : 0, 1)$: In this case, $l = m = 1$. Again we have $t(-r_0-r_1) > (d-1)(-r_0-r_1) \geq -(d-2)r_0-dr_1$. Now, if $t(-r_0-r_1) \leq -dr_0-(d-2)r_1$, then $(d-\frac{4}{3})(-r_0-r_1) < -dr_0-(d-2)r_1$, so $r_1 > 2r_0$. This means that $t(-r_0 - r_1) < (d - \frac{4}{3})(-r_0 - r_1) < -(d - 4)r_0 - dr_1$. It follows that $a_{0,0} = a_{1,0} = 0$, and either $a_{0,1} = 0$ or $a_{2,0} = 0$. But we know that exactly one of the two lines through p intersects C with multiplicity exactly 2, so neither of these is a possibility.

Case 2 – p lies on the line $y_0 = 0$, but not the line $x_0 = 0$: Following the same argument as above we see that either p lies on a linear component of C, or every point of C that lies on the line $x_0 = 0$ also lies on the line $y_0 = 0$. It follows that $(p, C) \notin X^-$, a contradiction.

Case 3 – p lies on the line $x_0 = 0$, but not on the line $y_0 = 0$: Again, following the same argument as above we see that p lies on a linear component of C. This implies that $(p, C) \notin X^-$, which is impossible.

Case 4 – p does not lie on either of the lines $x_0 = 0$ or $y_0 = 0$: In this case, $l = m = 0$. As above, we see that $a_{k_0,k_1} = 0$ if k_0 and k_1 are both less than d, and $a_{d,k} = 0$ for $k < d - 1$. Now, if $(d - \frac{3}{2})(r_0 + r_1) \leq (d - 6)r_0 + dr_1$, then $3r_0 \leq r_1$, so $t(r_0 + r_1) > (d - \frac{3}{2})(r_0 + r_1) \geq dr_0 + (d - 2)r_0$. It follows that either $a_{d,d-1} = 0$, in which case C is a union of $2d$ lines, which is impossible, or $a_{k,d} = 0$ for all $k < d - 2$. We therefore see that C is of the form

$$C = x_0^{d-2} y_0^{d-1}(a_{d,d}x_0^2 y_0 + a_{d,d-1}x_0^2 y_1 + a_{d-1,d}x_0 x_1 y_0 + a_{d-2,d}x_1^2 y_0).$$

Thus, C consists of three components. One is a curve of bidegree $(2, 1)$. The other two components consist of multiple lines through one of the points on this curve that has a tangent line. The point p is forced to be the unique other such point.

It is clear that this $(p, C) \in X^-$, since by definition, one of the lines through p intersects C with multiplicity greater than 1, and it is impossible for it to intersect a smooth curve of bidegree $(2,1)$ with higher multiplicity than 2, or for the other line through p to intersect the curve with multiplicity at all. To see that $(p, C) \notin X^{ss}(t_0 - \epsilon)$, consider the 1-parameter subgroup with weights $(-1, 1, -2, 2)$.

Finally, notice that all such curves are in the same orbit of the action of G, so X^0 must be the set of *all* such curves. To see this, note that if we fix the two points that have tangent lines to be $(1,0 : 1,0)$ and $(0,1 : 0,1)$, then the curve is determined uniquely by the third point of intersection of the curve with the diagonal. Since $PSL(2,\mathbb{C})$ acts 3-transitively on points of $\mathbb{P}^1$, we obtain the desired result.

□

Corollary 5.6. *The map $X//_{t_0-\epsilon}G \to X//_{t_0}G$ contracts the locus $H_{0,1} \backslash H_{1,1}$ to a point. Outside of this locus, the map is an isomorphism.*

Proof. Let $C = x_1^{d-2} y_1^{d-1}(x_0^2 y_1 + x_1^2 y_0)$, and $p = (0,1 : 0,1)$. Then $(p, C) \in X^0$. As we have seen, X^0 is the orbit of (p, C), so $G \cdot (p, C)$ is closed in $X^{ss}(t_0)$ and $X^0//_{t_0}G$ is a point. Notice that the stabilizer of (p, C) must fix $p = (0,1 : 0,1)$, and the other ramification point, which is $(1,0 : 1,0)$. Thus, the stabilizer of (p, C) must consist solely of pairs of diagonal matrices. A quick check shows that the stabilizer of (p, C) is the one-parameter subgroup with weights $(-1, 1, -2, 2)$, which is isomorphic to $\mathbb{C}^*$. Again, the conclusion follows from Theorem 2.2.

□

Our main interest is the case where $d = 3$. As above, this is because in this case $X//_{t_0-\epsilon}G$ is a birational model for $\overline{M}_{4,1}$. In particular, we have the following:

Proposition 5.7. *There is a birational contraction $\beta : \overline{M}_{4,1} \dashrightarrow X//_{t_0-\epsilon}G$.*

Proof. As above, it suffices to exhibit a morphism $\beta^{-1} : V \to \overline{M}_{4,1}$, where $V \subseteq X//_{t_0-\epsilon}G$ is open with complement of codimension ≥ 2 and β^{-1} is an isomorphism onto its image. Again, we let $U \subseteq X^{ss}(t_0 - \epsilon)$ be the set of all moduli stable pointed curves $(p, C) \in X^{ss}(t_0 - \epsilon)$. The proof in this case is exactly like that in the case of $\mathbb{P}^2$, as the discriminant locus $\Delta \subseteq X$ is again an irreducible G-invariant hypersurface.

By the universal property of the moduli space, since U is a family of moduli stable curves, it admits a unique map $U \to \overline{M}_{4,1}$. Since U is contained in the semistable locus and this map is G-equivariant, it factors uniquely through a map $U//_{t_0-\epsilon}G \to \overline{M}_{4,1}$. Since every curve of bidegree $(3,3)$ on $\mathbb{P}^1 \times \mathbb{P}^1$ is canonical, two such curves are isomorphic if and only if they differ by an automorphism of $\mathbb{P}^1 \times \mathbb{P}^1$. It follows that this map is an isomorphism onto its image.

□

Theorem 5.8. *There is a birational contraction of $\overline{M}_{4,1}$ contracting the pointed Brill-Noether divisor $BN^1_{3,(0,2)}$. Moreover, if P is the Petri divisor, then the divisors $BN^1_{3,(0,2)}$, P, Δ_1, Δ_2, and Δ_3 span a simplicial face of $\overline{NE}^1(\overline{M}_{4,1})$.*

Proof. The composition $\overline{M}_{4,1} \dashrightarrow X//_{t_0-\epsilon}G \to X//_{t_0}G$ is a birational contraction. By the above, the given pointed Brill-Noether divisor is contracted by this map. It therefore suffices to show that the isomorphism β^{-1} constructed in the preceding theorem does not contain in its image the generic point of P or Δ_i for $i \geq 1$. Every smooth curve in X is Gieseker-Petri general, since its canonical embedding lies on a smooth quadric, so the generic point of P is not contained in the image of β^{-1}. For Δ_i this again follows directly from the fact that $\Delta \cap U$ is an irreducible divisor in U whose generic point is an irreducible nodal curve.

□

References

[Art09] Michela Artebani. A compactification of M_3 via $K3$ surfaces. *Nagoya Math. J.*, 196:1–26, 2009.

[CF91] Fernando Cukierman and Lung-Ying Fong. On higher Weierstrass points. *Duke Math. J.*, 62(1):179–203, 1991.

[CML09] Sebastian Casalaina-Martin and Radu Laza. The moduli space of cubic threefolds via degenerations of the intermediate Jacobian. *J. Reine Angew. Math.*, 633:29–65, 2009.

[Cuk97] Fernando Cukierman. Determinant of complexes and higher Hessians. *Math. Ann.*, 307(2):225–251, 1997.

[DH98] I. V. Dolgachev and Y. Hu. Variation of geometric invariant theory quotients. *Inst. Hautes Études Sci. Publ. Math.*, (87):5–56, 1998.

[EH87] David Eisenbud and Joe Harris. The Kodaira dimension of the moduli space of curves of genus ≥ 23. *Invent. Math.*, 90(2):359–387, 1987.

[Far09] Gavril Farkas. Koszul divisors on moduli spaces of curves. *Amer. J. Math.*, 131(3):819–867, 2009.

[FP05] Gavril Farkas and Mihnea Popa. Effective divisors on $\overline{M}_g$, curves on $K3$ surfaces, and the slope conjecture. *J. Algebraic Geom.*, 14(2):241–267, 2005.

[HH09a] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first divisorial contraction. *Trans. Amer. Math. Soc.*, 361(8):4471–4489, 2009.

[HH09b] Brendan Hassett and Donghoon Hyeon. Log canonical models for the moduli space of curves: the first flip. *preprint*, 2009.

[HL10] Donghoon Hyeon and Yongnam Lee. Log minimal model program for the moduli space of stable curves of genus three. *Math. Res. Lett.*, 17(4):625–636, 2010.

[HM82] Joe Harris and David Mumford. On the Kodaira dimension of the moduli space of curves. *Invent. Math.*, 67(1):23–88, 1982. With an appendix by William Fulton.

[Log03] Adam Logan. The Kodaira dimension of moduli spaces of curves with marked points. *Amer. J. Math.*, 125(1):105–138, 2003.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[Rul01] William Frederick Rulla. *The birational geometry of moduli space M(3) and moduli space M(2,1)*. ProQuest LLC, Ann Arbor, MI, 2001. Thesis (Ph.D.)–The University of Texas at Austin.

[Tha96] M. Thaddeus. Geometric invariant theory and flips. *J. Amer. Math. Soc.*, 9(3):691–723, 1996.