MA 665 EXERCISES 11

(1) Let $V \subseteq \mathbb{A}^n$, $W \subseteq \mathbb{A}^m$ be varieties. Shows that

$$V \times W := \{(a_1, \ldots, a_n, b_1, \ldots, b_m) | (a_1, \ldots, a_n) \in V, (b_1, \ldots, b_m) \in W\}$$

is a variety in $\mathbb{A}^{n+m}$.

(2) Let I be an ideal in a ring R. Show that if a^n and b^m are in I, then so is $(a+b)^{n+m}$. Conclude that

$$\text{Rad}(I) := \{a \in R | a^n \in I \text{ for some positive integer } n\}$$

is an ideal in R.

(3) Let $X \subset \mathbb{A}^n$ be a variety.

(a) Let $p \in \mathbb{A}^n$ be a point not in X. Show that there is a polynomial $f \in k[x_1, \ldots, x_n]$ such that $f(q) = 0$ for all $q \in X$, and $f(p) = 1$.

(b) Now let $p_1, \ldots, p_r \in \mathbb{A}^n$ be distinct points not in X. Show that there are polynomials $f_1, \ldots, f_r \in k[x_1, \ldots, x_n]$ such that $f_i(q) = 0$ for all $q \in X$, $f_i(p_j) = 0$ for all $j \neq i$, and $f_i(p_i) = 1$.

(c) Let $a_{ij} \in k$ be constants for $1 \leq i, j \leq r$. Show that there are polynomials $g_1, \ldots, g_r \in I(X)$ such that $g_i(p_j) = a_{ij}$ for all i and j.