MA 665 EXERCISES 4

(1) If R is an integral domain, the *rank* of an R-module M is the maximum number of R-linearly independent elements of M. Show that if R is an integral domain and I is any nonprincipal ideal of R, then I is torsion free of rank 1 but is not a free R-module.

(2) Let R be an integral domain with quotient field K and let M be any R-module. Prove that the rank of M is equal to the dimension of the K-vector space $K \otimes_R M$.

(3) Prove that a finitely generated module P over a PID is free if and only if it satisfies the following condition: for any surjective R-module homomorphism $\varphi : M \to N$ and any $f \in \mathrm{Hom}_R(P, N)$, there exists a “lift” $F \in \mathrm{Hom}_R(P, M)$ such that $f = \varphi \circ F$. (Modules that satisfy this condition are called *projective* modules.)