CHIP FIRING

14. Reduced Divisors on Metric Graphs

We now develop the theory of reduced divisors on metric graphs, analogous to the corresponding theory for discrete graphs. We begin with the definition.

Definition 14.1. *Let Γ be a metric graph and $v \in \Gamma$. A divisor D on Γ is v-*reduced *if*

(1) *D is effective away from v and*
(2) *every closed connected subset $A \subseteq \Gamma \backslash \{v\}$ contains a point x with $D(x) < \mathrm{outdeg}_A(x)$.*

This is reminiscent of the definition for discrete graphs, but with a set of vertices replaced with a connected closed set. As in the discrete case, we will show that every divisor on a metric graph is equivalent to a unique v-reduced divisor. In this way, v-reduced divisors give a natural choice of representatives for divisors classes on metric graphs. We begin by showing that v-reduced divisors, if they exist, are unique. The proof is almost verbatim the same as in the discrete case.

Theorem 14.2. *If $D \sim D'$ are v-reduced, then $D = D'$.*

Proof. By definition, there exists $\varphi \in \mathrm{PL}(\Gamma)$ such that $D' - D = \mathrm{div}(\varphi)$. If $D \neq D'$, then φ is not constant. Let $A \subset \Gamma$ be a connected component of the subset where φ achieves its maximum. Note that A is nonempty because Γ is compact, and it is not all of Γ because φ is not constant. By exchanging D and D' if necessary, we may assume that $v \notin A$.

Note that φ is constant on A and has positive incoming slope along any tangent vector leaving A. It follows that $\mathrm{ord}_w(\varphi) \geq \mathrm{outdeg}_A(w)$ for all $w \in A$. Since D is effective away from v, we have $D(w) \geq 0$ for all $w \in A$. Hence

$$D'(w) = D(w) + \mathrm{ord}_w(\varphi) \geq D(w) + \mathrm{outdeg}_A(w) \geq \mathrm{outdeg}_A(w)$$

for all $w \in A$. It follows that D' is not v-reduced. □

To prove the existence of v-reduced divisors, we first need the following lemma.

Lemma 14.3. *Let $D \in \mathrm{Div}(\Gamma)$, $v \in \Gamma$, and let G be a model for Γ with vertex set containing $\{v\} \cup \mathrm{supp}(D)$. Then D is v-reduced if and only if the corresponding divisor D on G is v-reduced.*

Proof. It is clear that D is effective away from v if and only if the corresponding divisor on G is effective away from v.

Now, let $A \subset \Gamma \smallsetminus \{v\}$ be a closed connected subset such that $D(x) \geq \mathrm{outdeg}_A(x)$ for all $x \in A$. Then the boundary of A is contained in the support of D, hence

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A is a union of vertices and edges in the model G. Writing $V(A)$ for the set of vertices in this set, we see that $\text{outdeg}_A(x) \geq \text{outdeg}_{V(A)}(x)$ for all $x \in V(A)$, so $D(x) \geq \text{outdeg}_{V(A)}(x)$ for all $x \in V(A)$. It follows that the corresponding divisor on G is not v-reduced.

Now, suppose that the corresponding divisor on G is not v-reduced. Then, run Dhar's burning algorithm and let $A \subset \Gamma$ be a connected component of the set of unburnt points. We then see that $D(x) \geq \text{outdeg}_A(x)$ for all $x \in A$, so D is not v-reduced. □

Remark 14.4. Note that Lemma 14.3 does not imply that the v-reduced divisor on Γ corresponds to the v-reduced divisor on G. In particular, since our choice of model G depends on the divisor D, replacing D with an equivalent divisor may force us to change the model.

We prove the existence of a v-reduced divisor equivalent to a given divisor D in two parts, starting with the case where D is effective. As in the case of discrete graphs, we produce an iterative procedure that at each step identifies a divisor equivalent to D that is "closer" to being v-reduced. This proof is a bit more delicate than the discrete case, because the linear series $|D|$ is infinite, so one must be careful to check that the procedure terminates.

Theorem 14.5. *If D is effective away from v, then D is equivalent to a v-reduced divisor.*

Proof. Let G be a model for Γ with vertex set containing v. Choose an ordering of $V(G) \cup E(G)$ such that every edge is adjacent to a vertex that precedes it, and every vertex other than v is adjacent to an edge that precedes it. This induces a quasi-order[1] on points of Γ, which in turn induces the lexicographic quasi-order on divisors of Γ.

Now, given $D \in \text{Div}(\Gamma)$ effective away from v, we will show that either D is v-reduced or we can find a divisor equivalent to D that is strictly larger in the lexicographic quasi-order. Since the model G has only finitely many vertices and edges, there are only finitely many equivalence classes of divisors under the equivalence induced by the quasi-order, so this process will eventually terminate.

Now, let G' be the refinement of G obtained by adding $\text{supp}(D)$ to the vertex set. Run Dhar's burning algorithm on G' and let $A \subset \Gamma$ be the set of unburnt points. If $A = \emptyset$, then by Lemma 14.3, D is v-reduced. Otherwise, let d be the minimal distance from A to $V(G)\backslash A$. Let χ be the rational function that takes the value d on all points outside a d-neighborhood of A, the value 0 at all points in A, and has slope 1 for a distance of d on the edges emanating from A. Then $D + \text{div}(\chi)$ is effective away from v, and is strictly larger in the lexicographic quasi-ordering on divisors. □

To handle the case where D is not effective, we will use the following result.

Corollary 14.6. *Any effective divisor of degree at least $g+r$ on a metric graph Γ of genus g has rank at least r.*

[1] A quasi-order is a reflexive, transitive binary relation such that, for any v, w, either $v \leq w$ or $w \leq v$ (or possibly both). It is like a total ordering, but distinct elements may have the same order.

Proof. We prove this by induction on r, the case $r = 0$ being obvious. Suppose that $r \geq 1$. We first show that every effective divisor D of degree $g+r$ fails to be v-reduced for $v \in \Gamma \smallsetminus \operatorname{supp}(D)$. To see this, let G be a model for Γ with vertex set containing $\{v\} \cup \operatorname{supp}(D)$. By Riemann-Roch, the corresponding divisor on G has rank at least r, hence it is not v-reduced. By Lemma 14.3, it follows that D is not v-reduced.

By Theorem 14.5, if D is an effective divisor of degree $g + r$, then there exists a v-reduced divisor D_v equivalent to D, which by the above must contain v in its support. It follows that, for any point $v \in \Gamma$, there exists a divisor $D_v \sim D$ such that $D_v - v$ is an effective divisor of degree $g + r - 1$. By induction, $r(D - v) \geq r - 1$ for all $v \in \Gamma$, and thus $r(D) \geq r$. □

We now complete the proof that every divisor on a metric graph is equivalent to a unique v-reduced divisor. Recall that, in the discrete case, we used the fact that the Jacobian is finite, and hence every element of the Jacobian is torsion, to complete the proof. This argument will not work in the case of metric graphs, where the Jacobian has non-torsion elements.

Lemma 14.7. *Every divisor on Γ is equivalent to a divisor that is effective away from v.*

Proof. Let $D \in \operatorname{Div}(\Gamma)$. Write $D = E - E'$, with E and E' effective. By Corollary 14.6, $(g+\deg(E'))v$ has rank at least $\deg(E')$, hence $(g+\deg(E'))v-E'$ is equivalent to an effective divisor. It follows that

$$D = E + ((g + \deg(E'))v - E') - (g + \deg(E'))v$$

is equivalent to a divisor that is effective away from v. □

Lemma 14.7 allows us to remove the effective hypothesis from Corollary 14.6.

Corollary 14.8. *Every divisor class of degree at least g on a metric graph of genus g is effective.*

Proof. Let D be a divisor of degree at least g on Γ and let $v \in \Gamma$. By Lemma 14.7, there exists an effective divisor E such that $D \sim E - (\deg(E) - \deg(D))v$. By Corollary 14.6, the rank of E is at least $\deg(E) - g \geq \deg(E) - \deg(D)$, hence $E - (\deg(E) - \deg(D))v$ is equivalent to an effective divisor. □

Corollary 14.9. *Every divisor class of degree at least $g + r$ on a metric graph of genus g has rank at least r.*

Proof. This is an immediate consequence of Corollaries 14.8 and 14.6. □