CHIP FIRING

16. Divisor Theory of Weighted Graphs

Our goal for this section is to prove a structure theorem for Jacobians of metric graphs. This will require a study of the various different models of a given metric graph. If Γ is a metric graph and G is a model for Γ, then the edge lengths of G determine a function $\ell : E(G) \to \mathbb{R}_{>0}$. We may think of the pair (G, ℓ) as a *weighted graph.*

Definition 16.1. *Let A be a ring. The* module of 0-chains *on G with coefficients in A, denoted $C_0(G, A)$, is the free A-module on the vertices of G. The* module of 1-chains *on G, denoted $C_1(G, A)$, is the free A-module on the edges of G.*

Throughout this section, the ring A will be either $\mathbb{Z}$ or $\mathbb{R}$. Note that $C_0(G, \mathbb{Z})$ is the same as $\mathrm{Div}(G)$. Given an orientation on G, we define a map $d : C_0(G, \mathbb{R}) \to C_1(G, \mathbb{R})$ by

$$df(e) := \frac{f(e^+) - f(e^-)}{\ell(e)},$$

where e^+ denotes the head of e and e^- denotes the tail of e. We also define a map $d^* : C_1(G, A) \to C_0(G, A)$ by

$$d * \alpha(v) := \sum_{e^+=v} \alpha(e) - \sum_{e^-=v} \alpha(e).$$

Example 16.2. Let G be a graph with all edge lengths 1. The module $C_0(G, \mathbb{R})$ is generated by the functions f_v given by

$$f_v(w) = \begin{cases} 1 & \text{if } w = v \\ 0 & \text{if } w \neq v. \end{cases}$$

Given a vertex v, choose an orientation of G so that all edges adjacent to v are directed toward v. Then

$$df_v(e) = \begin{cases} 1 & \text{if } e \text{ is adjacent to } v \\ 0 & \text{otherwise.} \end{cases}$$

We then see that

$$d^* df(w) = \begin{cases} \mathrm{val}(v) & \text{if } w = v \\ -1 & \text{if } w \text{ is adjacent to } v \\ 0 & \text{otherwise.} \end{cases}$$

In other words, $d^* df_v$ is the negative of the divisor obtained, starting with the zero divisor, by firing the vertex v. Since the functions f_v generate $C_0(G, \mathbb{R})$, we see that $d^*(\mathrm{Im}(d) \cap C_1(G, \mathbb{Z}))$ is precisely the set of divisors equivalent to 0.

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We now generalize Example 16.2 to weighted graphs with more general edge lengths.

Definition 16.3. *Let* (G, ℓ) *be a weighted graph. We define the set of* principal divisors *on* (G, ℓ) *to be*

$$\mathrm{Prin}(G, \ell) := d^*(\mathrm{Im}(d) \cap C_1(G, \mathbb{Z})).$$

The Jacobian *of the weighted graph* (G, ℓ) *is defined to be*

$$\mathrm{Jac}(G, \ell) := \mathrm{Div}^0(G)/\mathrm{Prin}(G, \ell).$$

Example 16.4. Consider the cycle graph G with two vertices v and w, connected by two edges. Suppose that the total length of the cycle is 1, so if the length of one edge is r, then the length of the other edge is $1-r$. We will compute the Jacobian of this weighted graph. Note that $\mathrm{Div}^0(G)$ is isomorphic to $\mathbb{Z}$, generated by the divisor $v - w$.

If we orient the graph so that both edges are directed toward v, then df_v takes the value $\frac{1}{r}$ on one edge and $\frac{1}{1-r}$ on the other edge. Note that $df_w = -df_v$, so $\mathrm{Im}(d)$ is generated by df_v. Therefore, to compute $\mathrm{Im}(d) \cap C_1(G, \mathbb{Z})$, we must classify those real numbers t such that both $\frac{t}{r}$ and $\frac{t}{1-r}$ are integers.

We consider two cases. If r is rational, write $r = \frac{m}{n}$, where m and n are relatively prime. In order for $\frac{t}{r}$ to be an integer, t must be rational as well, so write $t = \frac{a}{b}$, where a and b are relatively prime. If $\frac{t}{r} = \frac{an}{bm}$ is an integer, then b must divide an, but since a and b are relatively prime, b must divide n. It follows that t is of the form $t = \frac{a}{n}$, where a is an integer. Since $\frac{t}{r} = \frac{a}{m}$ is an integer, we see that m must divide a, and since $\frac{t}{1-r} = \frac{a}{n-m}$, we see that $n-m$ must divide a. Since m and $n-m$ are relatively prime, we see that $m(n-m)$ must divide a. Thus, $\mathrm{Im}(d) \cap C_1(G, \mathbb{Z})$ is generated by the 1-chain α that takes the value m on one edge and $n-m$ on the other edge. We therefore see that $\mathrm{Prin}(G, \ell)$ is generated by $d^*\alpha$, which takes the value n at v and $-n$ at w. Thus,

$$\mathrm{Jac}(G, \ell) = \mathbb{Z}/n\mathbb{Z}.$$

On the other hand, if r is irrational, and $\frac{t}{r}, \frac{t}{1-r}$ are both integers, then $t = 0$. It follows that, in this case, $\mathrm{Prin}(G, \ell) = 0$. Thus,

$$\mathrm{Jac}(G, \ell) = \mathbb{Z}.$$