CHIP FIRING

4. Dhar's Burning Algorithm

Last time, we saw that every divisor class contains a unique v-reduced representative. In this lecture, we describe an algorithm for computing this representative. This procedure is known as Dhar's Burning Algorithm.

Given a divisor D and a vertex v, we may compute the v-reduced divisor equivalent to D as follows:

(1) Replace D with an equivalent divisor that is effective away from v, using the technique of Lemma 3.3above.
(2) Start a fire at v.
(3) Burn every edge of the graph that is adjacent to a burnt vertex.
(4) Running through the vertices in lexicographic order, burn the first vertex w with the property that the number of burnt edges adjacent to w exceeds the number of chips $D(w)$ at w. If no such vertex exists, proceed to step (5). Otherwise, return to step (3).
(5) Let A be the set of unburnt vertices. If A is non-empty, then $D + D_A$ is effective away from v. Replace D with $D + D_A$ and return to step (2). Otherwise, if A is empty, then D is v-reduced.

Proof of Dhar's Burning Algorithm. To see that the algorithm works, note that by construction $D(w) \geq \text{outdeg}_w(A)$ for all $w \in A$. It follows that $D + D_A$ is effective away from v. Conversely, if A' is a set of vertices such that $D + D_{A'}$ is effective away from v for some $A' \subset V(G) \smallsetminus \{v\}$, then A' consist only of unburnt vertices. That is, the set of unburnt vertices is lexicographically minimal amongst all sets A' such that $D + D_{A'}$ is effective away from v. To see this, let $w \in A'$ be the first vertex in A' that burns. By construction, this implies that $D(w) < \text{outdeg}_w(A')$, hence $D + D_{A'}$ is not effective away from v. The algorithm terminates for the same reason as in Proposition 3.5 – there are only finitely many divisors of given degree that are greater than a given divisor in the lexicographic ordering. $\square$

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Example 4.1. We run Dhar's Burning Algorithm to compute the v-reduced divisor equivalent to the divisor pictured in Figure 1, where v is the lower right vertex.

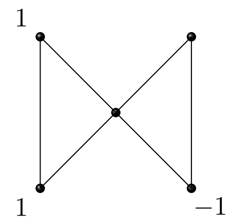

FIGURE 1. A divisor that is effective away from v

After burning vertices and edges until there are none left to burn, we see that the two vertices on the left remain unburnt. Therefore, the divisor pictured in Figure 2 is not v-reduced.

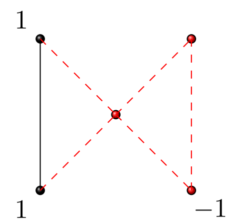

FIGURE 2. First iteration of Dhar's Burning Algorithm

Firing the two vertices on the left, we obtain the divisor pictured in Figure 3.

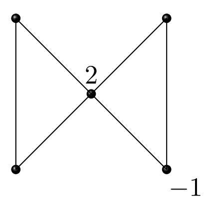

FIGURE 3. Result of firing the left two vertices

Running a second iteration of Dhar's Burning algorithm, we see that the three left vertices are unburnt, as in Figure 4.

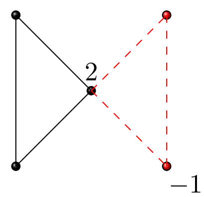

FIGURE 4. Second iteration of Dhar's Burning Algorithm

Firing these three vertices, we obtain the divisor pictured in Figure 5.

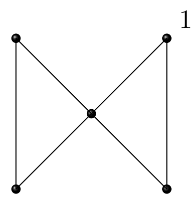

FIGURE 5. Result of firing the left three vertices

Running Dhar's Burning Algorithm a final time, we see that the whole graph burns. Therefore, the divisor depicted in Figure 5 is v-reduced.

The importance of Dhar's Burning Algorithm is that it allows us to determine whether a given divisor class contains an effective representative.

Proposition 4.2. *A divisor class $[D]$ has an effective representative if and only if its v-reduced representative is effective.*

Proof. If the v-reduced representative is effective, then clearly there exists an effective representative. The converse follows from Dhar's burning algorithm; if we start with an effective representative of the divisor class and run Dhar's algorithm, then the divisor stays effective at every step. □