CHIP FIRING

5. Orientable Divisors

Definition 5.1. *The* genus *of a graph G is*
$$g = |E(G)| - |V(G)| + 1.$$

The genus of a graph is its first Betti number. In other words, it is the rank of $H^1(G, \mathbb{Z})$. We use the term genus to emphasize the analogy between divisors on graphs and divisors on algebraic curves. This should not be confused with the other common graph invariant known as the genus, which is the minimal genus of a surface in which the graph can be embedded without crossings.

Another invariant of a graph is its canonical divisor, defined as follows.

Definition 5.2. *The canonical divisor of a graph G is the divisor*
$$K_G = \sum_{v \in V(G)} (\mathrm{val}(v) - 2)v.$$

Much of the theory we have developed concerning divisors on graphs can also be formulated in terms of graph orientations. The connection between divisors and graph orientations begins with orientable divisors.

Definition 5.3. *Let G be a graph and $\mathcal{O}$ an orientation of G. The corresponding* orientable divisor *is*
$$D_{\mathcal{O}} := \sum_{v \in V(G)} (\mathrm{indeg}_{\mathcal{O}}(v) - 1)v.$$

Note that, if $\mathcal{O}$ is an orientation and $\overline{\mathcal{O}}$ is the reverse orientation, then
$$D_{\mathcal{O}} + D_{\overline{\mathcal{O}}} = K_G.$$
Another simple fact about orientable divisors is that they all have the same degree.

Lemma 5.4. *Let G be a graph of genus g. Every orientable divisor on G has degree $g - 1$.*

Proof. Let $\mathcal{O}$ be an orientation of G. Then
$$\deg D_{\mathcal{O}} = \sum_{v \in V(G)} (\mathrm{indeg}_{\mathcal{O}}(v) - 1) = \sum_{v \in V(G)} (\mathrm{indeg}_{\mathcal{O}}(v)) - |V(G)|.$$
Each edge in G contributes one to the sum $\sum_{v \in V(G)} (\mathrm{indeg}_{\mathcal{O}}(v))$, so the expression above equals
$$|E(G)| - |V(G)| = g - 1.$$
□

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As a consequence, we see that the canonical divisor has degree $2g-2$.

A key connection between chip firing and graph orientations is the following observation.

Proposition 5.5. *Let G be a graph and $\mathcal{O}$ an orientation of G. Let $A \subset V(G)$ be a subset of the vertices with the property that all edges in the cut (A, A^c) are directed toward A, and let $\mathcal{O}'$ be the orientation obtained from $\mathcal{O}$ by reversing this directed cut. Then*

$$D_{\mathcal{O}'} = D_{\mathcal{O}} + D_A.$$

In particular, $D_{\mathcal{O}'}$ is equivalent to $D_{\mathcal{O}}$.

Proof. For every edge in the cut (A, A^c), the divisor $D_{\mathcal{O}'}$ has one fewer chip than $D_{\mathcal{O}}$ at the edge's endpoint in A, and one more chip than $D_{\mathcal{O}}$ at the edge's endpoint in A^c. Thus, replacing $D_{\mathcal{O}}$ with $D_{\mathcal{O}'}$ has the same effect as firing the set A. $\square$

Definition 5.6. *Let G be a graph and v a vertex of G. An orientation $\mathcal{O}$ of G is called* v-connected *if, for every vertex w in G, there is a directed path in $\mathcal{O}$ from v to w.*

Corollary 5.7. *Let G be a graph and $\mathcal{O}$ an orientation of G. There exists a v-connected orientation $\mathcal{O}'$ such that $D_{\mathcal{O}} \sim D_{\mathcal{O}'}$.*

Proof. Let $A \subseteq V(G)$ be the set of vertices that can be reached from v by a directed path in $\mathcal{O}$. We proceed by induction on $|A^c|$. If $A = V(G)$, then $\mathcal{O}$ is v-connected, and we are done. Now, suppose that $A \neq V(G)$. By definition, the cut (A, A^c) is directed toward A. Let $\mathcal{O}'$ be the orientation obtained from $\mathcal{O}$ by reversing this cut. By Proposition 5.5, $D_{\mathcal{O}'}$ is equivalent to $D_{\mathcal{O}}$. If $A' \subseteq V(G)$ is the set of vertices that can be reached by a directed path in $\mathcal{O}'$, then A is strictly contained in A'. By induction, therefore, there is a v-connected orientation $\mathcal{O}''$ such that $D_{\mathcal{O}'}$ is equivalent to $D_{\mathcal{O}''}$, and the result follows. $\square$

Theorem 5.8. *Let G be a graph of genus g. Every divisor on G of degree $g-1$ is equivalent to an orientable divisor.*

Proof. Let D be a divisor on G of degree $g-1$, and let $\mathcal{O}$ be an orientation of G. We provide an algorithm in which, at each step, we either modify the orientation $\mathcal{O}$ or replace D with an equivalent divisor. When the algorithm terminates, we will have $D = D_{\mathcal{O}}$.

Let

$$E^+ = \sum_{v \in V(G), D(v) - D_{\mathcal{O}}(v) > 0} (D(v) - D_{\mathcal{O}}(v))v$$

and

$$E^- = \sum_{v \in V(G), D(v) - D_{\mathcal{O}}(v) < 0} (-D(v) + D_{\mathcal{O}}(v))v.$$

Note that both E^+ and E^- are effective, $D - D_{\mathcal{O}} = E^+ - E^-$, and since D and $D_{\mathcal{O}}$ have the same degree, $\deg E^+ = \deg E^-$.

We proceed by induction on $\deg E^+$. If $\deg E^+ = 0$, then since E^+ is effective, this implies that $E^+ = 0$. Since $\deg E^+ = \deg E^-$, we see that $E^- = 0$ as well. Thus, in this case, we have $D - D_{\mathcal{O}} = 0$, and the result follows.

Now, suppose that $\deg E^+ > 0$, and let $A \subseteq V(G)$ denote the set of vertices that can be reached from the support of E^+ by a directed path in $\mathcal{O}$. There are two cases to consider. First, if A contains a vertex in the support of E^-, then by definition there exists a directed path from a vertex v in the support of E^+ to a vertex w in the support of E^-. In this case, replace the orientation $\mathcal{O}$ with the orientation $\mathcal{O}'$ obtained by reversing this directed path. We then see that

$$D_{\mathcal{O}'} = D_{\mathcal{O}} + v - w,$$

so

$$D - D_{\mathcal{O}'} = D - D_{\mathcal{O}} - v + w = (E^+ - v) - (E^- - w).$$

Since v is in the support of E^+, we see that this operation decreases $\deg E^+$ by 1, and the result follows by induction.

Finally, suppose that A does not contain a vertex in the support of E^-. We proceed by induction on $|A^c|$. If $A = V(G)$, then it clearly contains such a vertex, so we may assume that $A \neq V(G)$. Then, by definition, the cut (A, A^c) is directed towards A. In this case, replace the orientation $\mathcal{O}$ with the orientation $\mathcal{O}'$ obtained by reversing this directed path, and replace D with the equivalent divisor $D + D_A$. By Proposition 5.5, we have

$$D + D_A - D_{\mathcal{O}'} = D - D_{\mathcal{O}}.$$

If A' is the set of vertices that can be reached from the support of E^+ by a directed path in $\mathcal{O}'$, then A is strictly contained in A', and the result follows by induction. □

Theorem 5.8 has the following interesting consequence.

Corollary 5.9. *Let G be a graph of genus g. Every divisor of degree at least g on G is equivalent to an effective divisor.*

Proof. Let D be a divisor on G of degree $d \geq g$. Let v be a vertex of G. The divisor $D - (d - g + 1)v$ has degree $g - 1$, so by Theorem 5.8, there exists an orientation $\mathcal{O}$ such that

$$D_{\mathcal{O}} \sim D - (d - g + 1)v.$$

Moreover, by Corollary 5.7, we may assume that $\mathcal{O}$ is v-connected.

We now show that the divisor $D_{\mathcal{O}} + (d - g + 1)v$ is effective. Note that $\operatorname{indeg}_{\mathcal{O}}(v) \geq 0$, so

$$D_{\mathcal{O}}(v) + d - g + 1 \geq -1 + d - g + 1 \geq 0.$$

For any vertex $w \neq v$, since $\mathcal{O}$ is v-connected, we have $\operatorname{indeg}_{\mathcal{O}}(w) \geq 1$, so

$$D_{\mathcal{O}}(w) \geq 0.$$

Therefore, $D_{\mathcal{O}} + (d - g + 1)v$ is effective. □