CHIP FIRING

9. Riemann-Roch

Perhaps the most important theorem concerning ranks of divisors is the Riemann-Roch theorem.

Theorem 9.1. *Let D be a divisor on a graph G of genus g. Then*

$$\mathrm{rk}(D) - \mathrm{rk}(K_G - D) = \deg(D) - g + 1.$$

Proof. Our alternate characterization of the rank gives us

$$\begin{aligned}&\mathrm{rk}(D) - \mathrm{rk}(K_G - D)\\ &= \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(K_G - D' - D_{\mathcal{O}})\}.\end{aligned}$$

Recall that, if $\mathcal{O}$ is an orientation and $\overline{\mathcal{O}}$ is the reverse orientation, then $D_{\mathcal{O}} + D_{\overline{\mathcal{O}}} = K_G$. Moreover, $\overline{\mathcal{O}}$ is acyclic if and only if $\mathcal{O}$ is acyclic. We may therefore rewrite the expression above as

$$\begin{aligned}&\mathrm{rk}(D) - \mathrm{rk}(K_G - D)\\ &= \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D_{\mathcal{O}} - D')\}.\end{aligned}$$

For any divisor E, we have $\deg(E) = \deg^+(E) - \deg^+(-E)$. We may use this to rewrite the expression $\deg^+(D_{\mathcal{O}} - D')$ in terms of $\deg^+(D' - (D_{\mathcal{O}})$, as follows.

$$\begin{aligned}&\mathrm{rk}(D) - \mathrm{rk}(K_G - D)\\ &= \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}}) - \deg(D' - D_{\mathcal{O}})\}\\ &= \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}})\} - \min_{\substack{D' \sim D\\ \mathcal{O} \text{ acyclic}}} \{\deg^+(D' - D_{\mathcal{O}}) - (\deg(D) - (g-1))\}\\ &= \deg(D) - g + 1.\end{aligned}$$

□

We now explore some consequences of the Riemann-Roch theorem.

Corollary 9.2. *Let D be a divisor on a graph of genus g. If $\deg(D) > 2g - 2$, then $\mathrm{rk}(D) = \deg(D) - g$.*

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Proof. If $\deg(D) > 2g-2$, then $\deg(K_G - D) < 0$. Since a divisor of negative degree cannot be effective, we see that $\mathrm{rk}(K_G - D) = -1$. By Riemann-Roch, we then have

$$\mathrm{rk}(D) + 1 = \deg(D) - g + 1,$$

so

$$\mathrm{rk}(D) = \deg(D) - g.$$

□

By Corollary 9.2, if a divisor has large degree, then its rank is completely determined by its degree. Similarly, if a divisor has negative degree, then it has rank -1. It follows that, on a given graph G, there are only finitely many divisors whose rank is not determined by their degree. In the edge cases, when the degree of a divisor is 0 or $2g-2$, there are two possibilities.

Corollary 9.3. *Let D be a divisor on a graph of genus g. If $\deg(D) = 2g-2$, then*

$$\mathrm{rk}(D) = \begin{cases} g-1 & \textit{if } D \sim K_G \\ g-2 & \textit{otherwise.} \end{cases}$$

Proof. If $\deg(D) = 2g-2$, then $\deg(K_G - D) = 0$. Since the only effective divisor of degree 0 is the zero divisor, we see that $K_G - D$ has rank 0 if and only if it is equivalent to 0, and rank -1 otherwise. By Riemann-Roch, we then have that D has rank $g-1$ if and only if it is equivalent to K_G, and rank $g-2$ otherwise. □

Our next consequence of Riemann-Roch is usually referred to as the Clifford bound.

Theorem 9.4. *Let D be a divisor on a graph G, and suppose that both D and $K_G - D$ have nonnegative rank. Then*

$$\mathrm{rk}(D) \leq \frac{1}{2}\deg(D).$$

Proof. By Riemann-Roch, we have

$$\mathrm{rk}(D) - \mathrm{rk}(K_G - D) = \deg(D) - g + 1.$$

By subadditivity of the rank, we have

$$\mathrm{rk}(D) + \mathrm{rk}(K_G - D) \leq \mathrm{rk}(K_G) = g - 1.$$

Adding these together, we obtain

$$2\,\mathrm{rk}(D) \leq \deg(D).$$

□

As mentioned above, there are only finitely many divisors on a given graph G whose rank is not determined by their degree. For each such divisor, we see that

$$\max\{-1, \deg(D) - g\} \leq \mathrm{rk}(D) \leq \frac{1}{2}\deg(D).$$

There are therefore only finitely many possibilities for the rank of such a divisor. The possible pairs (d, r), where d and r are the degree and rank, respectively, of a divisor, are illustrated in Figure 1.

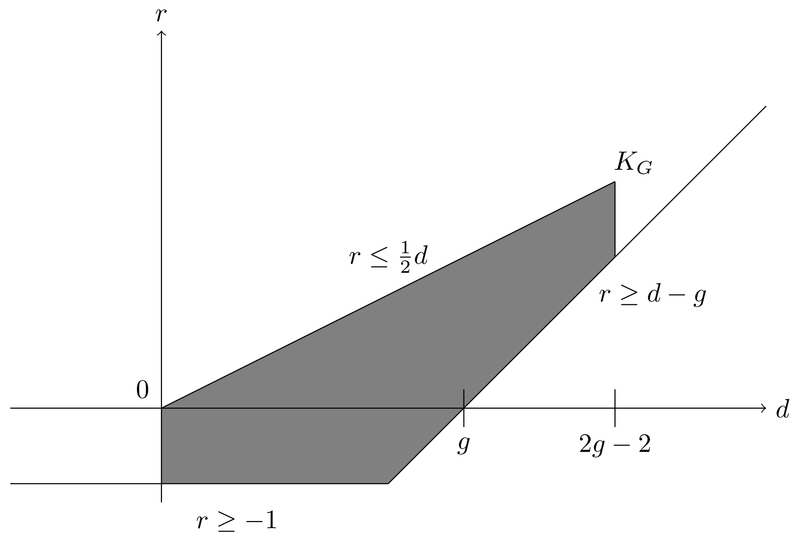

FIGURE 1. Possibilities for the degree and rank of a divisor.

Of particular interest are the divisors that have larger than expected rank. These divisors correspond to lattice points in Figure 1 that lie above the lower bound. Figure 1 shows that a graph has only finitely many such divisors.

Definition 9.5. *A divisor D on a graph of genus g is called* special *if*

$$\mathrm{rk}(D) > \max\{-1, \deg(D) - g\}.$$

Example 9.6. A graph of genus zero is a tree. If D is a divisor on a tree and $\deg(D) < 0$, then $\mathrm{rk}(D) = -1$. On the other hand, if $\deg(D) > 2g - 2 = -2$, then $\mathrm{rk}(D) = \deg(D) - g = \deg(D)$. It follows that there are no special divisors on a tree.

Example 9.7. A cycle has genus 1. If D is a divisor on a cycle and $\deg(D) < 0$, then $\mathrm{rk}(D) = -1$. On the other hand, if $\deg(D) > 2g - 2 = 0$, then $\mathrm{rk}(D) = \deg(D) - g = \deg(D) - 1$. It remains to consider divisors of degree zero. A divisor of degree zero is effective if and only if it is the zero divisor. Therefore, if $\deg(D) = 0$, then

$$\mathrm{rk}(D) = \begin{cases} 0 & \text{if } D \sim 0 \\ -1 & \text{otherwise.} \end{cases}$$

In other words, the only special divisor on a cycle is the zero divisor. Note that the canonical divisor of a cycle is the zero divisor.

Example 9.8. Consider a graph G of genus 2. If D is a divisor on G and $\deg(D) < 0$, then $\mathrm{rk}(D) = -1$. On the other hand, if $\deg(D) > 2g - 2 = 2$, then $\mathrm{rk}(D) = \deg(D) - 2$. As in Example 9.7, if $\deg(D) = 0$, then $\mathrm{rk}(D) = 0$ if and only if $D \sim 0$, and $\mathrm{rk}(D) = -1$ otherwise. By Corollary 9.3, if $\deg(D) = 2$, then $\mathrm{rk}(D) = 1$ if and only if $D \sim K_G$, and $\mathrm{rk}(D) = 0$ otherwise.

It remains to consider divisors of degree 1. By the Clifford bound, we see that the rank of a divisor of degree 1 is either 0 or -1. We can in fact characterize these divisors as follows. Let v be a vertex of G. For any divisor D of degree 1 on G, there exists a v-connected orientation $\mathcal{O}$ such that $D \sim D_{\mathcal{O}}$, and the rank of D is -1 if and only if $\mathcal{O}$ is acyclic. Figure 2 depicts all the v-connected orientable divisors on a graph of genus 2, where v is the vertex in the lower left. The first 4 orientations are acyclic, so the corresponding orientable divisor has rank -1. The remaining 5 orientations contain a directed cycle, so the corresponding orientable divisor has rank zero. The three divisors in the bottom row are evidently effective. We invite the reader to use Dhar's Burning Algorithm to check that the second two divisors in the middle row are equivalent to effective divisors.

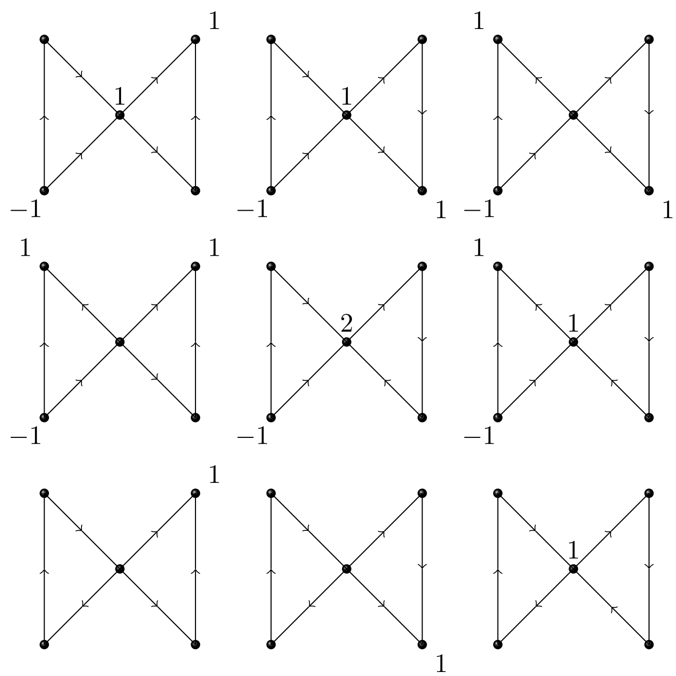

FIGURE 2. Orientable divisors on a graph of genus 2.