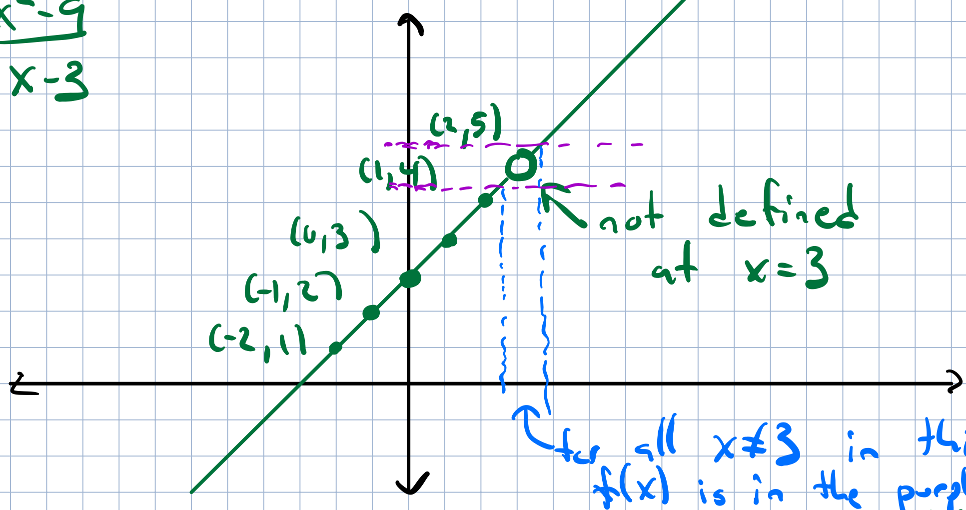
Limits of Functions

Ex: $f(x) = \dfrac{x^2-9}{x-3}$ ← Not defined when $x=3$.

$y = \dfrac{x^2-9}{x-3}$



If x is very close to 3, then y is very close to 6.

$$f(x) = \frac{x^2-9}{x-3} = \frac{(x+3)(x-3)}{x-3}$$

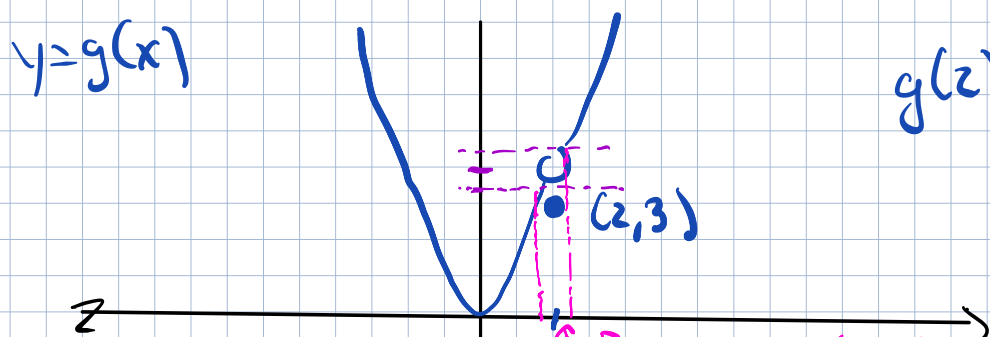
If $x \neq 3$, you can cancel the $x-3$ from the numerator & denominator, and we see that

$$f(x) = x+3 \quad \text{if } x \neq 3.$$

Not defined when $x=3$, but if x is <u>close</u> to 3, then y is <u>close</u> to 6.

We say that the limit of $f(x)$ as x approaches 3 is 6 and we write $\lim\limits_{x \to 3} f(x) = 6$.

Ex: $g(x) = \begin{cases} x^2 & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$



$g(2) = 3$.

For every $x \neq 2$ in this pink region, $g(x)$ is in the purple region.

If you plug in a value of x that is close to, but not equal to, 2 then the value of y is close to 4.

Even though $g(2) = 3$, the limit of $g(x)$ as x approaches 2 is 4.

$$\lim_{x \to 2} g(x) = 4.$$

"Def": Let $f(x)$ be a function. If, when you plug in values of x that are close to, but not equal to a, the values of $f(x)$ get arbitrarily close to L, then we say that the limit of $f(x)$ as x approaches a is L, and we write

$$\lim_{x\to a} f(x) = L.$$

Def: If, no matter how close you want you to be to L, there is a distance from a such that for all $x \neq a$ within that distance from a, $f(x)$ is within the given distance from L, then $\lim_{x\to a} f(x) = L$.

WARNING! Not all functions have limits!

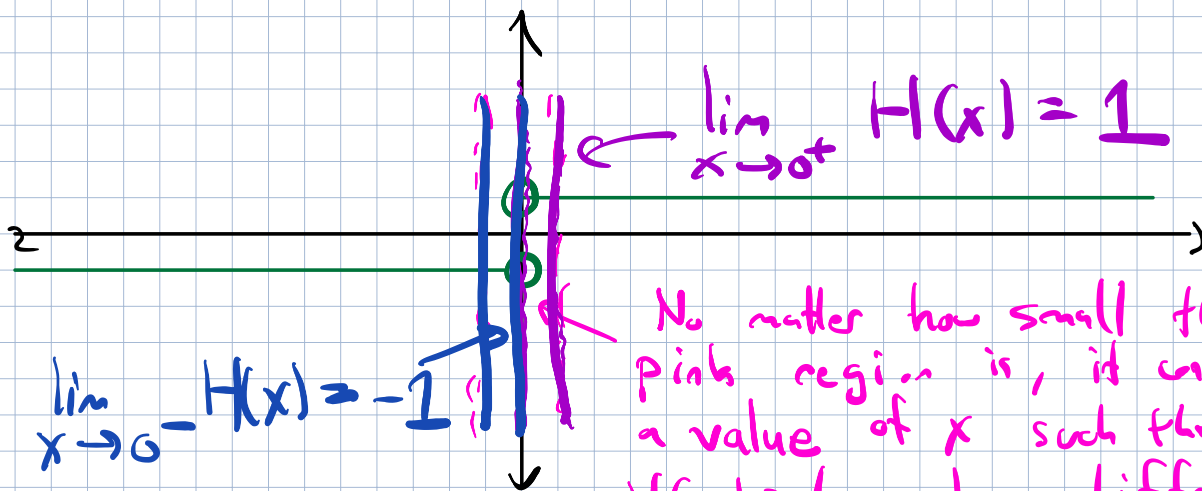
Ex: $H(x) = \dfrac{|x|}{x}$. Not defined when $x=0$.

What is $\lim_{x\to 0} H(x)$?

Recall $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$.

$$H(x) = \begin{cases} \frac{x}{x} & \text{if } x \geq 0 \\ \frac{-x}{x} & \text{if } x < 0 \end{cases}$$

$$= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}.$$



$\lim_{x \to 0^+} H(x) = 1$

$\lim_{x \to 0^-} H(x) = -1$

No matter how small this pink region is, it contains a value of x such that $H(x) = 1$ and a different value of x such that $H(x) = -1$.

So the values of $H(x)$ are <u>NOT</u> getting closer to any specific value.

$\lim_{x \to 0} H(x)$ does not exist.

<u>Def:</u> If, when you plug in values of x that are close to a and <u>greater than</u> a, the values of $f(x)$ get close to L, we write

$$\lim_{x \to a^+} f(x) = L.$$

If, when you plug in values of x that are close to a and <u>smaller than</u> a, the values of f(x) get close to L, we write

$$\lim_{x \to a^-} f(x) = L.$$

$\lim_{x \to a} f(x)$ exists if and only if

$\lim_{x \to a^-} f(x)$ exists and $\lim_{x \to a^+} f(x)$ exists

and $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$.

Ex: $f(x) = \sin\left(\frac{\pi}{x}\right)$.

Not defined when $x = 0$.

What is $\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$?

$x = 1$ $\quad \sin\left(\frac{\pi}{1}\right) = \sin(\pi) = 0$

$x = \frac{1}{2}$ $\quad \sin\left(\frac{\pi}{\frac{1}{2}}\right) = \sin(2\pi) = 0$

$x = \frac{1}{3}$ $\quad \sin\left(\frac{\pi}{\frac{1}{3}}\right) = \sin(3\pi) = 0$

$x = \frac{1}{4}$ $\quad \sin\left(\frac{\pi}{\frac{1}{4}}\right) = \sin(4\pi) = 0$

$x = 2 \qquad \sin\left(\frac{\pi}{2}\right) = 1$

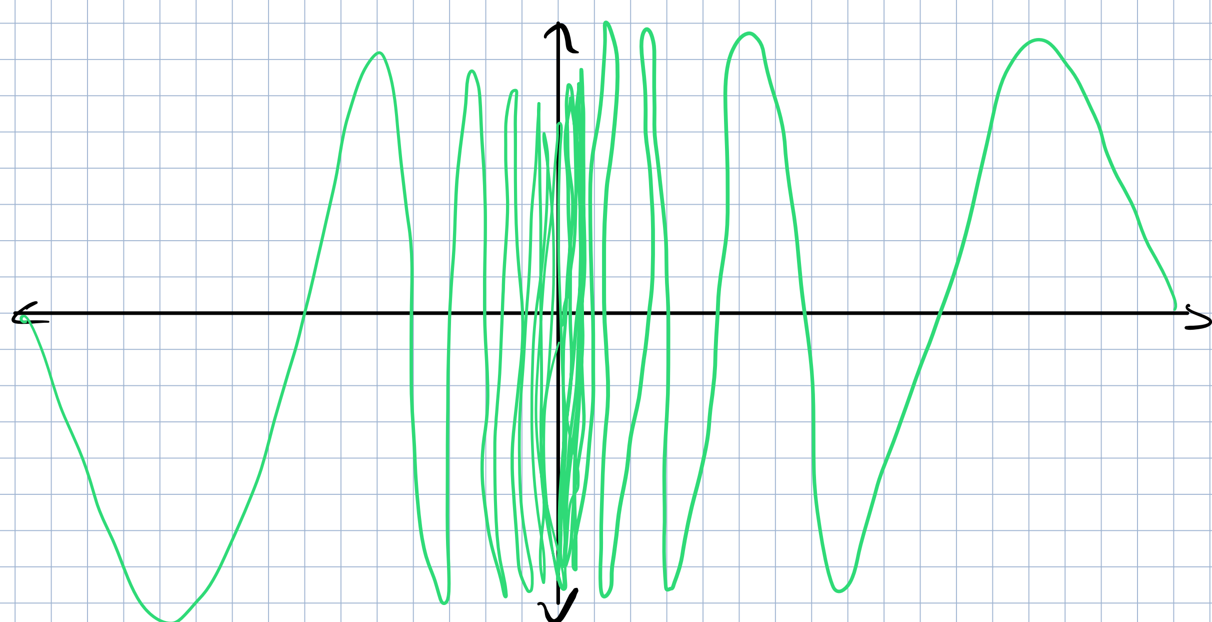
$x = \frac{2}{5} \qquad \sin\left(\frac{\pi}{\frac{2}{5}}\right) = \sin\left(\frac{5\pi}{2}\right) = 1$

$x = \frac{2}{9} \qquad \sin\left(\frac{\pi}{\frac{2}{9}}\right) = \sin\left(\frac{9\pi}{2}\right) = 1$

$x = \frac{2}{13} \qquad \sin\left(\frac{\pi}{\frac{2}{13}}\right) = \sin\left(\frac{13\pi}{2}\right) = 1$

$x = \frac{2}{3} \qquad \sin\left(\frac{\pi}{\frac{2}{3}}\right) = \sin\left(\frac{3\pi}{2}\right) = -1$

$x = \frac{2}{7} \qquad \sin\left(\frac{\pi}{\frac{2}{7}}\right) = \sin\left(\frac{7\pi}{2}\right) = -1$



No matter close you are to 0, there is a value of x closer to 0 with

$\sin\left(\frac{\pi}{x}\right) = 1$ and there is a value of x closer to 0 with $\sin\left(\frac{\pi}{x}\right) = -1$ (and everything in between.)

Because these values are not getting closer to any specific number, we say

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) \text{ does not exist.}$$