Chain Rule

$$\boxed{\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)}$$

Alternative Formulation:
If y is a function of x
and z is a function of y

$$\frac{dz}{dx} = \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Ex: Compute $\frac{d}{dx}\left[(3x+1)^{100}\right]$.

inside function $\quad g(x) = 3x+1$

outside function $\quad f(x) = x^{100}$

$$f(g(x)) = f(3x+1) = (3x+1)^{100}$$

$f(x) = x^{100}$ $\qquad$ $g(x) = 3x+1$

$f'(x) = 100x^{99}$ $\qquad$ $g'(x) = 3$

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)) \cdot g'(x)$$

$$= 100 \cdot g(x)^{99} \cdot 3 \quad = 100 \cdot (3x+1)^{99} \cdot 3$$

$$= 300 \cdot (3x+1)^{99}$$

Ex: Compute $\frac{d}{dx}\left[\sqrt{\frac{x+1}{x-1}}\right]$.

outside function is $f(x) = \sqrt{x}$

inside function is $g(x) = \frac{x+1}{x-1}$.

$$f(g(x)) = \sqrt{\frac{x+1}{x-1}}$$

$$f(x) = \sqrt{x}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

$$g(x) = \frac{x+1}{x-1}$$

$$g'(x) = \frac{(x-1)\frac{d}{dx}[x+1] - (x+1)\frac{d}{dx}[x-1]}{(x-1)^2}$$

QUOTIENT RULE

$$= \frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2}$$

$$= \frac{-2}{(x-1)^2}$$

$$\frac{d}{dx}\left[\sqrt{\frac{x+1}{x-1}}\right] \overset{\text{CHAIN RULE}}{=} f'(g(x))\cdot g'(x)$$

$$= \frac{1}{2\sqrt{g(x)}} \cdot \left(\frac{-2}{(x-1)^2}\right)$$

$$= \frac{1}{\cancel{2}\sqrt{\frac{x+1}{x-1}}} \cdot \left(\frac{-\cancel{2}}{(x-1)^2}\right)$$

$$= \frac{-\sqrt{\frac{x-1}{x+1}}}{(x-1)^2} = \frac{-\sqrt{\frac{1}{x+1}}}{(x-1)^{3/2}}$$

$$= \frac{-1}{(x+1)^{1/2}(x-1)^{3/2}}$$

Ex: Compute $\frac{d}{dx}\left[3(x^2+5x)^2 + 2(x^2+5x)+7\right]$

without the chain rule

$$=\frac{d}{dx}\left[3x^4+30x^3+75x^2+2x^2+10x+7\right]$$

$$=3\cdot 4x^3 + 30\cdot 3x^2 + 75\cdot 2x + 2\cdot 2x + 10$$

$$\boxed{=12x^3+90x^2+154x+10}$$

with the chain rule

inside function is $g(x)=x^2+5x$

outside function is $f(x)=3x^2+2x+7$

$$\frac{d}{dx}\left[3(x^2+5x)^2+2(x^2+5x)+7\right]=f'(g(x))\cdot g'(x)$$

$$=\left(6(x^2+5x)+2\right)\cdot(2x+5)$$

$$=(6x^2+30x+2)\cdot(2x+5)$$

$$=12x^3+60x^2+30x^2+150x+4x+10$$

$$\boxed{=12x^3+90x^2+154x+10}$$

Ex: Compute $\frac{d}{dx}\left[\sqrt{\sqrt{7x^2+5}+1}\right]$

outside function $f(x)=\sqrt{x}$

inside function $g(x)=\sqrt{7x^2+5}+1$

$f(x) = \sqrt{x}$

$f'(x) = \frac{1}{2\sqrt{x}}$

$g(x) = \sqrt{7x^2+5} + 1$

$g'(x)$ is itself a composition of functions, so to compute it, we need to use the chain rule AGAIN.

$f_1(x) = \sqrt{x} + 1$

$g_1(x) = 7x^2+5$

$f_1'(x) = \frac{1}{2\sqrt{x}}$

$g_1'(x) = 14x$

$g(x) = f_1(g_1(x))$ so by the CHAIN RULE,

$$g'(x) = f_1'(g_1(x)) \cdot g_1'(x)$$

$$= \frac{1}{2\sqrt{7x^2+5}} \cdot 14x = \frac{7x}{\sqrt{7x^2+5}}$$

By the chain rule AGAIN,

$$\frac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{2\sqrt{\sqrt{7x^2+5}+1}} \cdot \frac{7x}{\sqrt{7x^2+5}}$$

Iterated Chain Rule

$$\frac{d}{dx}\left[f(g(h(x)))\right] = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

Alternative Formulation

If y is a function of x and
z is a function of y and
w is a function of z, then

$$\frac{dw}{dx} = \frac{dw}{dz} \cdot \frac{dz}{dy} \cdot \frac{dy}{dx}$$

Higher Derivatives

If $f(x)$ is a function, the derivative of the derivative of $f(x)$ is called the **second derivative** of $f(x)$, and written

$$f''(x)$$

or $f^{(2)}(x)$.

The derivative of that is called the **third derivative**, and written $f'''(x)$ or $f^{(3)}(x)$.

AND SO ON.

Ex: Compute the 2^{nd} derivative of $f(x) = 7x^2 + 3x + 1$

$$f'(x) = 14x + 3$$

$$f''(x) = 14$$

$$f'''(x) = 0$$

If $f(x)$ = distance travelled by time x

$f'(x)$ = rate at which distance is changing at time x

= velocity at time x

$f''(x)$ = rate at which velocity at time x

= accelleration at time x