Final Project due Monday April 18
on Canvas

Intro to Differential Equations (MA 138, MA 214)

Last time we talked about antiderivatives.

Ex: Find a function whose derivative is

$$\frac{3}{x} - 5\sin(x).$$

We could write this as an equation:

$$y' = \frac{3}{x} - 5\sin(x).$$

This kind of equation is called a differential equation because it has a derivative in it.

Solve for y.

$$y = 3\cdot\ln(x) - 5\cdot(-\cos(x)) + C$$
$$= 3\ln(x) + 5\cos(x) + C$$

Solutions to differential equations are NOT necessarily unique!

A <u>differential equation</u> is an equation involving y'.

The usual goal is to solve for y.

A differential equation typically has more than 1 solution.

<u>Ex:</u> $y' = f(x)$.

A solution to this differential equation is a function y whose derivative is $f(x)$.

In other words, y is an antiderivative of $f(x)$.

<u>Ex:</u> $y' = y$.

$y = e^x$ is a solution to this diff. eq. because $y' = \frac{d}{dx}[e^x] = e^x = y$.

$y = 5 \cdot e^x$ is another solution, because

$$y' = \frac{d}{dx}[5e^x] = 5\frac{d}{dx}[e^x] = 5e^x = y$$

$y = k \cdot e^x$ where k is a constant is <u>also</u> a solution because

$$y' = \frac{d}{dx}[ke^x] = k \cdot \frac{d}{dx}[e^x] = ke^x = y$$

Are there any other solutions to

$$y' = y \quad ?$$

$$\frac{d}{dx}[\ln(y)] = \frac{y'}{y} = 1$$

$\ln(y)$ is an antiderivative of 1.

<u>All</u> antiderivatives of 1 are of the form:

$$x + C$$

$$\ln(y) = x + C$$

$$y = e^{x+C} = e^C \cdot e^x \qquad \text{write } k = e^C$$

$$= k \cdot e^x.$$

So <u>all</u> solutions to this diff. eq. are of the form $y = ke^x$.

Ex: Find all solutions to the diff. eq. $y' = y$ satisfying $y(0) = 17$.

This is called an initial value problem

We saw: all solns to the diff. eq. $y' = y$ are of the form $y = ke^x$.

Use the initial value to solve for k.

$$17 = y(0) = k \cdot e^0 = k \cdot 1 = k$$

$$k = 17$$

$$\boxed{y = 17e^x.}$$

Ex: Solve the IVP $y' = 2y$, $y(0) = 3$.

First, find <u>all</u> solutions to $y' = 2y$.

$$\frac{d}{dx}[\ln(y)] = \frac{y'}{y} = 2$$

$\ln(y)$ is an antiderivative of 2.

$$\ln(y) = 2x + C$$

$$y = e^{2x+c} = e^{c} \cdot e^{2x}$$ $k = e^{c}$

$$= k e^{2x}$$

Now plug in the initial value.

$$3 = y(0) = k \cdot e^{2 \cdot 0} = k e^{0} = k \cdot 1 = k$$

$$y = 3e^{2x}$$

Ex: Solve the diff. eq. $y' = my$ (m is a constant.)

$$\frac{d}{dx}[\ln(y)] = \frac{y'}{y} = m$$

$$\ln(y) = mx + c$$

$$y = e^{mx+c} = e^{c} \cdot e^{mx}$$

$$= k \cdot e^{mx}$$

Ex: $y'' = -y$.

$y = \sin(x)$ is a solution because

$y' = \frac{d}{dx}[\sin(x)] = \cos(x)$

$y'' = \frac{d}{dx}[\cos(x)] = -\sin(x) = -y$.

$y = a\cdot\sin(x)$ is also a solution because

$y' = \frac{d}{dx}[a\sin(x)] = a\frac{d}{dx}[\sin(x)] = a\cos(x)$

$y'' = \frac{d}{dx}[a\cos(x)] = a\frac{d}{dx}[\cos(x)] = -a\sin(x)$

$= -y$.

There are even more solutions!

$y = \cos(x)$ is also a solution

$y' = \frac{d}{dx}[\cos(x)] = -\sin(x)$

$y'' = \frac{d}{dx}[-\sin(x)] = -\cos(x) = -y$

For the same reason as above,
$y = b\cos(x)$ is also a solution.

$y = a\sin(x) + b\cos(x)$ is another solution.

$$y' = \frac{d}{dx}\left[a\sin(x) + b\cos(x)\right]$$
$$= a\cdot\cos(x) - b\sin(x)$$
$$y'' = \frac{d}{dx}\left[a\cos(x) - b\sin(x)\right]$$
$$= -a\sin(x) - b\cos(x) = -y$$

Notice: In this example, a single initial value is not enough to determine the solution.

Ex: $y'' = -y$, $y(0) = 7$.

$$y = a\sin(x) + b\cos(x)$$

$$7 = y(0) = a\sin(0) + b\cos(0)$$
$$= a\cdot 0 + b\cdot 1 = b$$

$b = 7$ but I don't know what a is!

$y = a\sin(x) + 7\cos(x)$ is a solution to this IVP for any number a.

Ex: $y'' = -y$, $y(0) = 7$, $y'(0) = 13$.

$y = a\sin(x) + 7\cos(x)$ is a soln.

$y' = a\cos(x) - 7\sin(x)$

$$13 = y'(0) = a\cos(0) - 7\sin(0)$$
$$= a \cdot 1 - 7 \cdot 0 = a.$$

$y = 13\sin(x) + 7\cos(x)$ is a soln to this IVP.