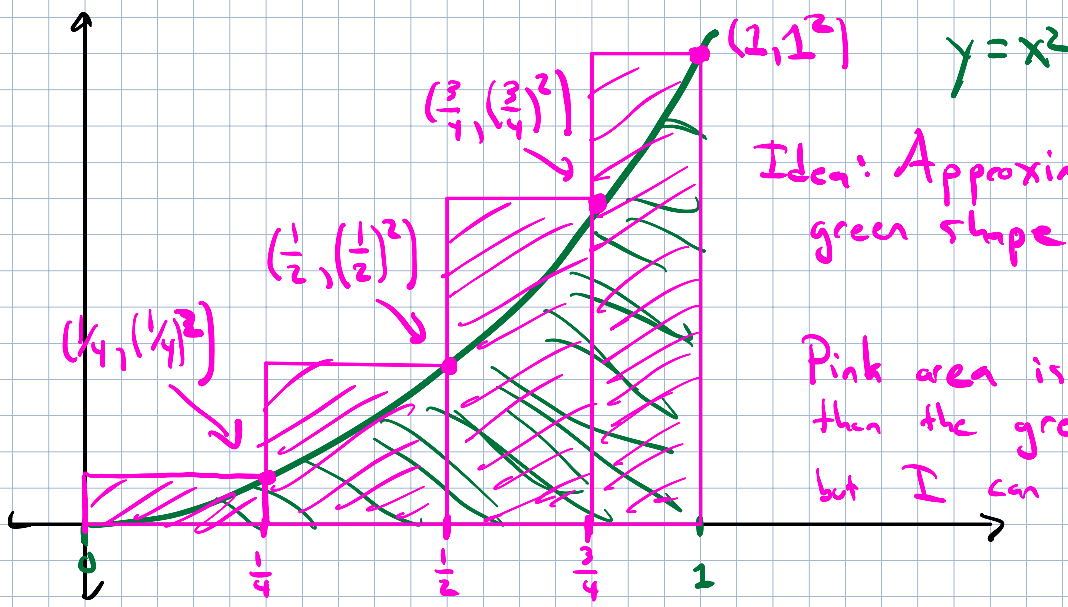
Final Exam Wednesday, May 4 10:30 AM – 12:30 PM

<u>here</u> (Chem-Phys 139)

It will be cumulative — that is, it will cover material from the entire semester

Area Problem

<u>Ex</u>: Compute the area under the parabola $y=x^2$ between $x=0$ and $x=1$.



Idea: Approximate the green shape with rectangles

Pink area is <u>larger</u> than the green area, but I can compute it!

Area of the 4 pink rectangles

$$= \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{2}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2 + \frac{1}{4}\left(\frac{4}{4}\right)^2$$

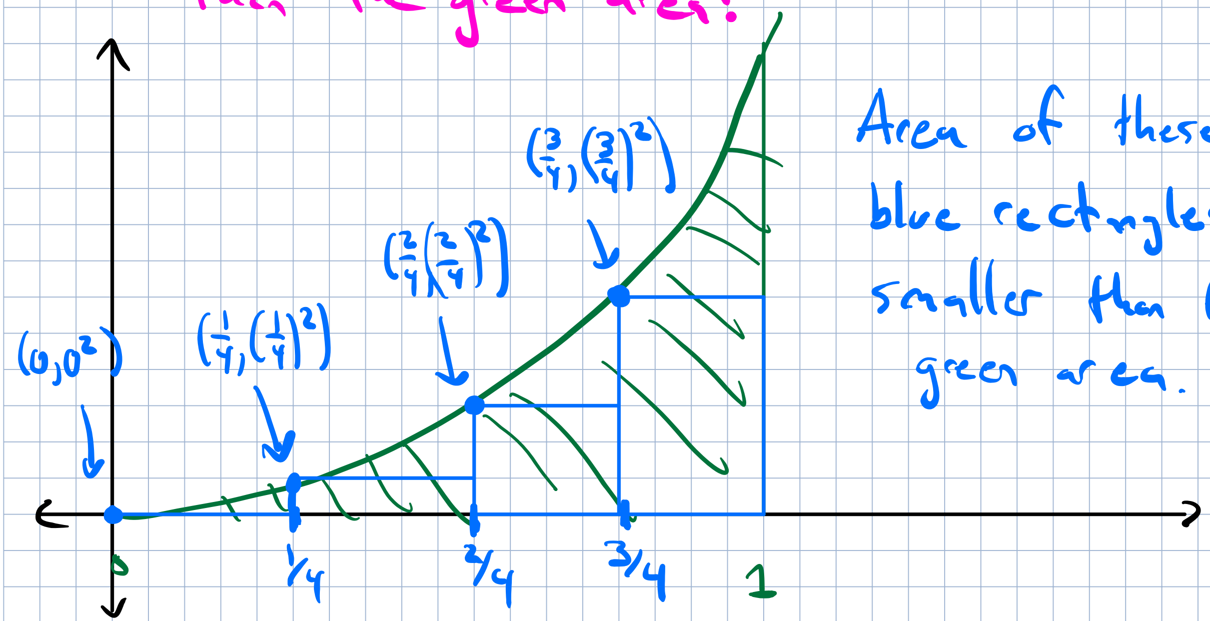
$$= \frac{1}{4}\cdot\left[\left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2 + \left(\frac{4}{4}\right)^2\right]$$

$$= \frac{1}{4^3}\cdot\left[1^2 + 2^2 + 3^2 + 4^2\right]$$

$$= \frac{1+4+9+16}{64} = \frac{30}{64} \approx 0.47$$

Recall: the green area is smaller than this!

What if we want a number that is smaller than the green area?



Area of these 4 blue rectangles is smaller than the green area.

Area of the 4 blue rectangles

$$= \frac{1}{4}\cdot 0 + \frac{1}{4}\cdot\left(\frac{1}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{2}{4}\right)^2 + \frac{1}{4}\cdot\left(\frac{3}{4}\right)^2$$

$$= \frac{1}{4}\left[\left(\frac{0}{4}\right)^2 + \left(\frac{1}{4}\right)^2 + \left(\frac{2}{4}\right)^2 + \left(\frac{3}{4}\right)^2\right]$$

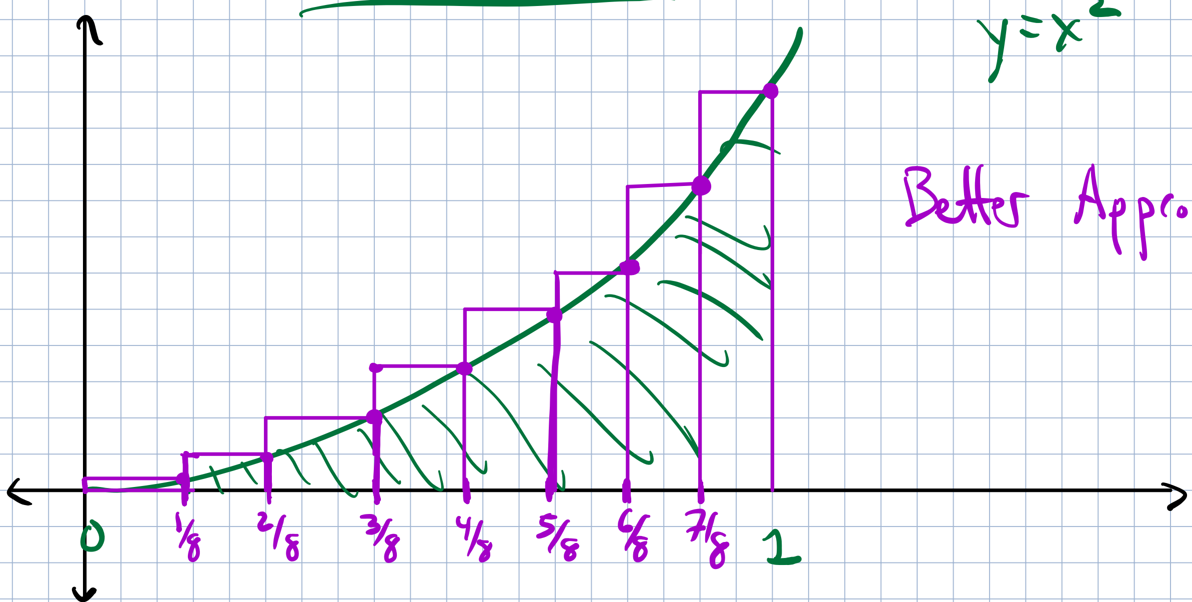
$$= \frac{1}{4^3}\left[0^2 + 1^2 + 2^2 + 3^2\right]$$

$$= \frac{0+1+4+9}{64} = \frac{14}{64} \approx 0.22$$

$0.22 \leq$ Green Area ≤ 0.47

What if we want a better approximation for the green area?

Idea: More Rectangles!



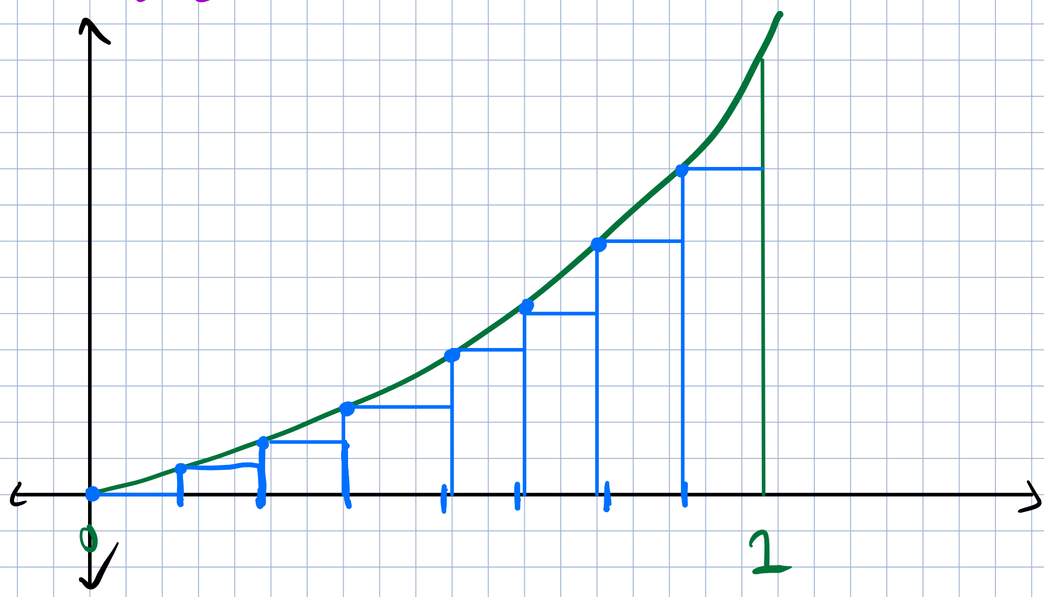
Area of the 8 purple rectangles:

$$= \frac{1}{8}\cdot\left(\frac{1}{8}\right)^2 + \frac{1}{8}\left(\frac{2}{8}\right)^2 + \frac{1}{8}\left(\frac{3}{8}\right)^2 + \frac{1}{8}\left(\frac{4}{8}\right)^2 + \frac{1}{8}\left(\frac{5}{8}\right)^2$$
$$+ \frac{1}{8}\left(\frac{6}{8}\right)^2 + \frac{1}{8}\left(\frac{7}{8}\right)^2 + \frac{1}{8}\left(\frac{8}{8}\right)^2$$

$$= \frac{1}{8^3}\cdot\left[1^2+2^2+3^2+4^2+5^2+6^2+7^2+8^2\right]$$

$$= \frac{1}{512}\left[1+4+9+16+25+36+49+64\right]$$

$= \frac{204}{512} \approx 0.39$



Area of these 8 blue rectangles

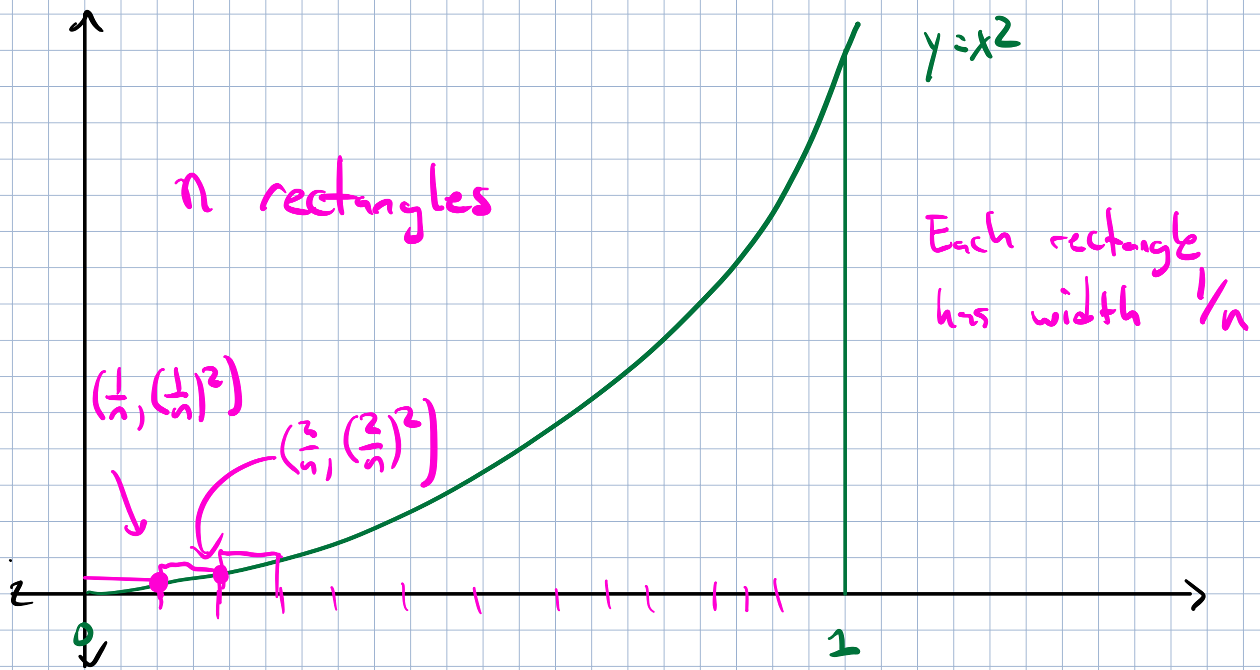
$$= \frac{1}{8^3}\left[0^2+1^2+2^2+3^2+4^2+5^2+6^2+7^2\right]$$

$$= \frac{140}{512} \approx 0.27$$

$$0.27 \le \text{Green Area} \le 0.39$$

If we want an even better approximation, take even more rectangles.

What if we have n rectangles?



Area of the n rectangles

$$= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \left(\frac{3}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right]$$

$$= \frac{1}{n^3}\left[1^2 + 2^2 + 3^2 + \cdots + n^2\right]$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6n^3}$$

To get the best possible approximation, take the limit as the number of rectangles goes to ∞

$$\lim_{n\to\infty} \frac{n(n+1)(2n+1)}{6n^3} = \frac{2}{6} = \frac{1}{3}.$$

If you do the same thing with left-justified rectangles instead, the rea of the n rectangles is

$$\frac{1}{n}\left[\left(\frac{0}{n}\right)^2+\left(\frac{1}{n}\right)^2+\left(\frac{2}{n}\right)^2+\cdots+\left(\frac{n-1}{n}\right)^2\right]$$

$$=\frac{1}{n^3}\left[0^2+1^2+2^2+\cdots(n-1)^2\right]$$

$$=\frac{(n-1)\cdot n\cdot(2n-1)}{6n^3}$$

$$\lim_{n\to\infty}\frac{(n-1)n(2n-1)}{6n^3}=\frac{2}{6}=\frac{1}{3}.$$

$$\frac{1}{3}\leq \text{Green Area} \leq \frac{1}{3}$$

$$\text{Green Area} = 1/3$$