MA 665 EXERCISES 3

(1) Let R be a ring and M an R-module. An element $m \in M$ is called a *torsion element* if $rm = 0$ for some nonzero $r \in R$. Prove that if R is an integral domain, then the set of torsion elements in M is a submodule of M. Give an example where R is not an integral domain, and the set of torsion elements in M is not a submodule of M.

(2) Let R be a commutative ring. Prove that M and $\mathrm{Hom}_R(R, M)$ are isomorphic as R-modules.

(3) Let R be a commutative ring and let A, B, M be R-modules. Prove that

$$\mathrm{Hom}_R(A \times B, M) \cong \mathrm{Hom}_R(A, M) \times \mathrm{Hom}_R(B, M)$$

and

$$\mathrm{Hom}_R(M, A \times B) \cong \mathrm{Hom}_R(M, A) \times \mathrm{Hom}_R(M, B).$$