

TROPICALIZATION OF THETA CHARACTERISTICS, DOUBLE COVERS, AND PRYM VARIETIES

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ABSTRACT. We study the behavior of theta characteristics on an algebraic curve under the specialization map to a tropical curve. We show that each effective theta characteristic on the tropical curve is the specialization of 2^{g-1} even theta characteristics and 2^{g-1} odd theta characteristics. We then study the relationship between unramified double covers of a tropical curve and its theta characteristics, and use this to define the tropical Prym variety.

1. INTRODUCTION

In this note we study the behavior of theta characteristics on an algebraic curve under the specialization map to a tropical curve. Throughout, we let Γ be a metric graph of genus g , k an algebraically closed nonarchimedean field of characteristic not equal to 2, and X an algebraic curve of genus g over k with skeleton Γ . We denote by $\Theta(X) \subset \text{Pic}(X)$ (respectively, $\Theta(\Gamma) \subset \text{Pic}(\Gamma)$) the set of theta characteristics on X (respectively, Γ). Our first main result is the following.

- Theorem 1.1.** (1) *The specialization map $\text{Trop} : \Theta(X) \rightarrow \Theta(\Gamma)$ is surjective. (Corollary 3.2.)*
- (2) *The preimage of the non-effective theta characteristic $L_0 \in \Theta(\Gamma)$ consists of 2^g even (in fact, non-effective) theta characteristics. (Lemma 4.2.)*
- (3) *The preimage of each other theta characteristic $L_\gamma \in \Theta(\Gamma)$ consists of 2^{g-1} even theta characteristics and 2^{g-1} odd theta characteristics. (Theorem 4.5.)*

The fact that there is precisely one non-effective theta characteristic on a metric graph is due to Zharkov [Zha10], whose characterization of $\Theta(\Gamma)$ we review in §4. The behavior of theta characteristics under specialization has received the attention of several authors. For example, in [BLM⁺16] it is shown that a tropical plane quartic admits 7 equivalence classes of bitangent lines, and it is conjectured that each one is the specialization of exactly 4 odd theta characteristics. This conjecture was established by Chan and Jiradilok in [CJ15] for tropical plane quartics of a certain combinatorial type, and for any combinatorial type in an upcoming paper by the second author and Markwig [LM16]. In the case where the curve X is hyperelliptic, Theorem 1.1 was proven by Panizzut in [Pan15].

Remark 1.2. Note that, although the parity of theta characteristics is preserved in flat families, it is not preserved by tropicalization. This is in contrast with the common slogan in tropical geometry which states that tropicalization “behaves like a flat family”. Nevertheless, Theorem 1.1 shows that tropicalization of theta characteristics is in some sense well-behaved.

The set $\Theta(X)$ is a torsor for the 2-torsion subgroup $\text{Jac}_2(X)$ of $\text{Jac}(X)$. On an algebraic curve, the set of 2-torsion divisor classes admits a nondegenerate pairing known as the Weil pairing. Our basic strategy for proving Theorem 1.1 is to study the relationship between this pairing and the specialization map. In particular, we show the following.

Theorem 1.3. *The kernel of the specialization map*

$$\text{Trop} : \text{Jac}_2(X) \rightarrow \text{Jac}_2(\Gamma)$$

is isotropic for the Weil pairing. (Proposition 4.3.)

After establishing Theorem 1.1, we then study the relationship between 2-torsion divisor classes on Γ and double covers of Γ . There is a natural bijection between $\text{Jac}_2(\Gamma)$ and the set of cycles in Γ . We define $D_\gamma \in \text{Jac}_2(\Gamma)$ to be the divisor class associated to a cycle γ . Classically, the set of 2-torsion divisor classes on an algebraic curve is in bijection with the set of unramified double covers of that curve. In the tropical setting, the analogue of a morphism between curves is a so-called harmonic morphism. In §5, we define the *dilation cycle* $\gamma(\varphi)$ of an unramified degree 2 harmonic morphism $\varphi : \tilde{\Gamma} \rightarrow \Gamma$, and show the following.

Theorem 1.4. *Let \mathcal{D} be a 2-torsion point in $\text{Jac}(X)$. Let $\bar{\varphi} : \tilde{X} \rightarrow X$ be the corresponding double cover and $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ the specialization of this cover. Then $\text{Trop}(\mathcal{D}) = D_{\gamma(\varphi)}$. (Theorem 5.11.)*

As a corollary, we give a completely combinatorial description of the full 2-torsion subgroup $\text{Jac}_2(X)$. Specifically, we see that $\text{Jac}_2(X)$ is an extension of a group, which is naturally identified with the set of degree 2 covering spaces of Γ , by $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$.

In §6, we define the Prym variety of an unramified degree 2 harmonic morphism. We use this to show how the Weil pairing of certain elements of $\text{Jac}_2(X)$ can be computed combinatorially, using only their specializations to the graph Γ .

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2. THE CLASSICAL THEORY OF THETA CHARACTERISTICS

In this section we review the classical theory of 2-torsion points on the Jacobian of a curve, and their relation to theta characteristics, double covers, and Prym varieties. All of the material of this section is standard, and can be found for example in [ACGH85, Appendix B] or [Har82].

2.1. Theta Characteristics and the Weil Pairing. A *theta characteristic* on X is a divisor class \mathcal{L} such that $2\mathcal{L} \sim K_X$. Throughout, we denote the set of theta characteristics on X by $\Theta(X)$. The set $\Theta(X)$ is a torsor for the 2-torsion subgroup $\text{Jac}_2(X)$. Since $\text{char}(k) \neq 2$, $\text{Jac}_2(X)$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{2g}$, so in particular $\Theta(X)$ has 2^{2g} elements.

A theta characteristic \mathcal{L} on X is called *even* if $h^0(X, \mathcal{L})$ is even, and *odd* if $h^0(X, \mathcal{L})$ is odd. Given a theta characteristic \mathcal{L} , one can define a quadratic form $q_{\mathcal{L}} : \text{Jac}_2(X) \rightarrow \mathbb{Z}/2\mathbb{Z}$ by

$$q_{\mathcal{L}}(\mathcal{D}) := h^0(X, \mathcal{L}) + h^0(X, \mathcal{L} + \mathcal{D}) \pmod{2}.$$

The associated bilinear pairing

$$\lambda(\mathcal{D}, \mathcal{E}) := q_{\mathcal{L}}(\mathcal{D}) + q_{\mathcal{L}}(\mathcal{E}) + q_{\mathcal{L}}(\mathcal{D} + \mathcal{E})$$

is independent of \mathcal{L} , and is known as the *Weil pairing*¹. Recall that a subspace $\Lambda \subset \text{Jac}_2(X)$ is called *isotropic* for a pairing λ if $\lambda(\Lambda, \Lambda) = 0$, and similarly, it is *isotropic* for a quadratic form q if $q|_{\Lambda}$ is identically zero.

For any nondegenerate bilinear form λ on a vector space over $\mathbb{Z}/2\mathbb{Z}$, there are exactly 2 quadratic forms with associated bilinear form λ . These are distinguished by the *Arf invariant*. The quadratic form with Arf invariant 1 has an isotropic subspace of dimension g , and has $(2^g + 1)2^{g-1}$ zeros. Conversely, the quadratic form with Arf invariant -1 does not have a g -dimensional isotropic subspace, and has $(2^g - 1)2^{g-1}$ zeros. One can show that the quadratic form $q_{\mathcal{L}}$ has Arf invariant 1 if and only if \mathcal{L} is even. It follows that $\Theta(X)$ contains exactly $(2^g + 1)2^{g-1}$ even theta characteristics and $(2^g - 1)2^{g-1}$ odd theta characteristics.

2.2. Double Covers and Prym Varieties. To any 2-torsion point $\mathcal{D} \in \text{Jac}_2(X)$, there exists a unique unramified double cover $\tilde{\varphi} : \tilde{X} \rightarrow X$ such that the kernel of the pullback map $\tilde{\varphi}^* : \text{Jac}(\tilde{X}) \rightarrow \text{Jac}(X)$ is $\{0, \mathcal{D}\}$. Indeed, if $2\mathcal{D} = \text{div}(f)$, then \tilde{X} is the curve with function field $K(X)(\sqrt{f})$. Conversely, given an unramified double cover one can recover the 2-torsion point \mathcal{D} by considering the kernel of the pullback map. In this way, there is a bijection between $\text{Jac}_2(X)$ and the set of unramified double covers of X .

Given such a double cover, the kernel of the pushforward map $\tilde{\varphi}_* : \text{Jac}(\tilde{X}) \rightarrow \text{Jac}(X)$ has two connected components. One defines the *Prym*

¹A more standard line of exposition would be to define the Weil pairing independently, and then deduce the relation above, often called the Riemann-Mumford relation. We will take this relation as the definition, as it is all that we need for our purposes.

variety $P(\bar{\varphi})$ to be the connected component of $\ker \bar{\varphi}_*$ containing 0. The Prym variety $P(\bar{\varphi})$ is an abelian variety of dimension $g - 1$.

3. JACOBIANS OF METRIC GRAPHS AND THEIR TORSION SUBGROUPS

Recall that the divisor group $\text{Div}(\Gamma)$ of a metric graph Γ is the free abelian group on points of the metric space Γ . A divisor $D = \sum a_i v_i$ on a metric graph is *effective* if $a_i \geq 0$ for all i . Its *degree* is defined to be

$$\deg(D) := \sum a_i.$$

A *rational function* on a metric graph Γ is a continuous, piecewise linear function $f : \Gamma \rightarrow \mathbb{R}$ with integer slopes. We write $\text{PL}(\Gamma)$ for the group of rational functions on Γ . Given $f \in \text{PL}(\Gamma)$ and $v \in \Gamma$, we define the *order of vanishing* of f at v , denoted $\text{ord}_v(f)$, to be the sum of the incoming slopes of f at v . The divisor associated to f is

$$\text{div}(f) = \sum_{v \in \Gamma} \text{ord}_v(f) \cdot [v].$$

Divisors of the form $\text{div}(f)$ are called *principal*.

We say that two divisors D and D' on a metric graph Γ are *equivalent* if $D - D'$ is principal. We define the *Picard group* of Γ to be the group of equivalence classes of divisors on Γ . That is,

$$\text{Pic}(\Gamma) = \text{Div}(\Gamma) / \text{div}(\text{PL}(\Gamma)).$$

The *Jacobian* $\text{Jac}(\Gamma)$ of Γ is the group of equivalence classes of divisors of degree zero.

The Jacobian $\text{Jac}(\Gamma)$ is a g -dimensional real torus, so its m -torsion subgroup is isomorphic to $(\mathbb{Z}/m\mathbb{Z})^g$. The main result of this section is the following.

Theorem 3.1. *The specialization map on m -torsion subgroups*

$$\text{Trop} : \text{Jac}_m(X) \rightarrow \text{Jac}_m(\Gamma)$$

is surjective.

Proof. Let A denote the value group of k . Note that since k is algebraically closed, the value group A is divisible. The universal cover of $\text{Jac}(X)^{\text{an}}$ is $(\mathbb{G}_m^{\text{an}})^g$. Let M' be the kernel of the map $(\mathbb{G}_m^{\text{an}})^g \rightarrow \text{Jac}(X)^{\text{an}}$, let M be the character lattice of \mathbb{G}_m^g , and let $N_A = \text{Hom}(M, A)$. By [BR15, (4.2.2)], there is a surjective homomorphism of short exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & M' & \longrightarrow & \mathbb{G}_m^g & \longrightarrow & \text{Jac}(X) \longrightarrow 0 \\ & & \downarrow \cong & & \downarrow & & \downarrow \text{Trop} \\ 0 & \longrightarrow & M' & \longrightarrow & N_A & \longrightarrow & \Sigma(\text{Jac}(X))_A \longrightarrow 0 \end{array}$$

By [BR15, Theorem 1.3], $\Sigma(\text{Jac}(X)) \cong \text{Jac}(\Gamma)$.

The preimage of the m -torsion subgroup $\text{Jac}_m(\Gamma)$ in $N_{\mathbb{R}}$ is the lattice $\frac{1}{m}M'$. Since A is divisible, N_A is as well, so $\frac{1}{m}M' \subset N_A$. For any $x \in \frac{1}{m}M'$, let $y \in M' \subset \mathbb{G}_m^g$ be the preimage of $mx \in M' \subset N_A$, and let $x' \in \mathbb{G}_m^g$ be any m th root of y . Then x' maps to x under the center vertical arrow, and maps to an m -torsion point in $\text{Jac}(X)$. It follows that the specialization map $\text{Trop} : \text{Jac}_m(X) \rightarrow \text{Jac}_m(\Gamma)$ is surjective. \square

Corollary 3.2. *The specialization map $\text{Trop} : \Theta(X) \rightarrow \Theta(\Gamma)$ is surjective.*

Proof. Let $\mathcal{L} \in \Theta(X)$ be a theta characteristic on X and $L \in \Theta(\Gamma)$ be a theta characteristic on Γ . Then $\text{Trop}(\mathcal{L}) - L \in \text{Jac}_2(\Gamma)$ is 2-torsion, hence by Theorem 3.1, there exists $\mathcal{D} \in \text{Jac}_2(X)$ such that $\text{Trop}(\mathcal{D}) = \text{Trop}(\mathcal{L}) - L$. Then $\mathcal{L} - \mathcal{D} \in \Theta(X)$ is a theta characteristic on X that specializes to L . \square

4. THETA CHARACTERISTICS

In this section, we consider the relationship between theta characteristics on the curve X and the metric graph Γ . The theta characteristics on a metric graph were characterized by Zharkov [Zha10]. We begin by describing his results.

First, there is a single non-effective theta characteristic, which can be obtained as follows. Let $v \in \Gamma$, and consider the distance function d_v whose value at a point $x \in \Gamma$ is the length of the shortest path from x to v . It is easy to see that $d_v \in \mathcal{L}(K_{\Gamma})$ is a piecewise linear function with all slopes of absolute value 1. If one orients Γ so that, with respect to the orientation, d_v has only positive slopes, then one obtains an orientation \mathcal{O} whose associated divisor

$$L_0 = \sum_{x \in \Gamma} (\text{indeg}_{\mathcal{O}}(x) - 1)x$$

has the property that $2L_0 = K_{\Gamma} + \text{div}(d_v)$. The class of this divisor is independent of the choice of point $v \in \Gamma$. Since \mathcal{O} is acyclic, one sees by [BN07, Lemma 3.2] or [MZ08, Lemma 7.8], that the divisor L_0 has negative rank. Indeed, the divisor L_0 is v -reduced and non-effective, so it is not equivalent to an effective divisor.

The remaining theta characteristics are all effective, and are in bijection with nonzero elements of $H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$. Let $\gamma \in H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$, $\gamma \neq 0$. By abuse of notation, we also denote by γ a cycle on Γ representing the class γ . Consider the distance function d_{γ} whose value at a point $x \in \Gamma$ is the length of the shortest path from x to the cycle γ . The function $d_{\gamma} \in \mathcal{L}(K_{\Gamma})$ is piecewise linear, with slope 0 on the cycle γ , and all other slopes of absolute value 1. If one orients Γ so that γ has a cyclic orientation and, everywhere else, d_{γ} has positive slopes with respect to the orientation, then one obtains an orientation \mathcal{O} whose associated divisor

$$L_{\gamma} = \sum_{x \in \Gamma} (\text{indeg}_{\mathcal{O}}(x) - 1)x$$

has the property that $2L_\gamma = K_\Gamma + \text{div}(d_\gamma)$. Note that L_γ is effective, and is supported on those points $x \in \Gamma$ for which the shortest distance from x to γ is obtained along at least two paths with distinct tangent directions at x .

Since $\Theta(\Gamma)$ contains a distinguished element L_0 , there is a canonical bijection between $\Theta(\Gamma)$ and $\text{Jac}_2(\Gamma)$. We define

$$(4.1) \quad D_\gamma := L_\gamma - L_0 \in \text{Jac}_2(\Gamma).$$

This is in contrast to the case of algebraic curves, where $\Theta(X)$ is a torsor for $\text{Jac}_2(X)$, but there is no distinguished element.

Example 4.1. We illustrate Zharkov’s construction with an example. Let Γ be the complete graph on 4 vertices, with all edge lengths 1. The 8 cycles on Γ can be partitioned into 3 types – the zero cycle, 4 “triangles”, and three “squares”. These cycles, together with the corresponding theta characteristics, are pictured in Figure 1. One can see by inspection that 7 of these theta characteristics are effective.

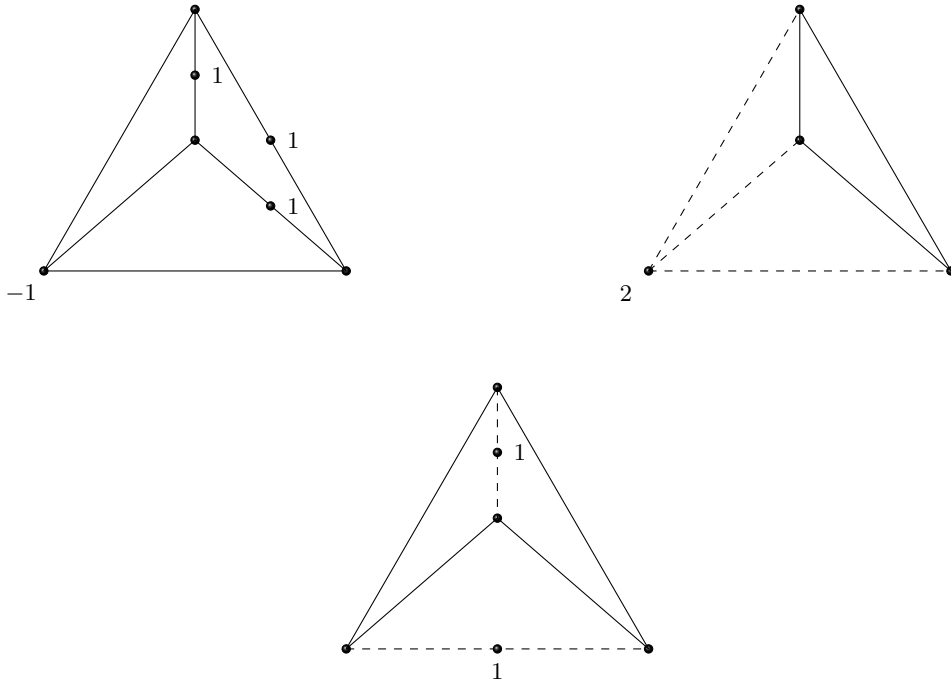


FIGURE 1. Three cycles on Γ with the corresponding theta characteristics

The following is an immediate consequence of Baker’s Specialization Lemma (see [Bak08, Lemma 2.8] or [AB15, Theorem 1.1]).

Lemma 4.2. *Let $\mathcal{L} \in \Theta(X)$ with $\text{Trop}(\mathcal{L}) = L_0$. Then \mathcal{L} is not effective.*

Proof. By Baker’s Specialization Lemma, $rk(\mathcal{L}) \leq rk(\text{Trop}(\mathcal{L}))$. But $\text{Trop}(\mathcal{L}) = L_0$ has rank -1, so \mathcal{L} has rank -1 as well. \square

For the remainder of the paper, we let Λ be the kernel of the specialization map $\text{Trop} : \text{Jac}_2(X) \rightarrow \text{Jac}_2(\Gamma)$. By Theorem 3.1, there is an exact sequence

$$0 \rightarrow \Lambda \rightarrow \text{Jac}_2(X) \rightarrow \text{Jac}_2(\Gamma) \rightarrow 0.$$

Thus, Λ is a g -dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. This space has an important property.

Proposition 4.3. *The subspace $\Lambda \in \text{Jac}_2(X)$ is isotropic for the Weil pairing.*

Proof. By Corollary 3.2, there exists $\mathcal{L} \in \Theta(X)$ such that $\text{Trop}(\mathcal{L}) = L_0$. For any $\mathcal{D} \in \Lambda$, by Lemma 4.2 we see that $h^0(X, \mathcal{L} + \mathcal{D}) = 0$. Hence, for any $\mathcal{D}, \mathcal{E} \in \Lambda$, we have

$$\begin{aligned} \lambda(\mathcal{D}, \mathcal{E}) &= q_{\mathcal{L}}(\mathcal{D}) + q_{\mathcal{L}}(\mathcal{E}) + q_{\mathcal{L}}(\mathcal{D} + \mathcal{E}) = \\ &= h^0(X, \mathcal{L} + \mathcal{D}) + h^0(X, \mathcal{L} + \mathcal{E}) + h^0(X, \mathcal{L} + \mathcal{D} + \mathcal{E}) = 0. \end{aligned}$$

□

Corollary 4.4. *For any $\mathcal{L} \in \Theta(X)$, the restriction of $q_{\mathcal{L}}$ to Λ is a linear map.*

Proof. Let $\mathcal{D}, \mathcal{E} \in \Lambda$. Then we have

$$\lambda(\mathcal{D}, \mathcal{E}) = q_{\mathcal{L}}(\mathcal{D}) + q_{\mathcal{L}}(\mathcal{E}) + q_{\mathcal{L}}(\mathcal{D} + \mathcal{E}).$$

By Proposition 4.3, however, the left hand side is zero. Hence, by rearranging the terms, we obtain

$$q_{\mathcal{L}}(\mathcal{D}) + q_{\mathcal{L}}(\mathcal{E}) = q_{\mathcal{L}}(\mathcal{D} + \mathcal{E}).$$

□

From this we deduce the main result.

Theorem 4.5. *For any effective class $L_{\gamma} \in \Theta(\Gamma)$, its preimage under the specialization map $\text{Trop} : \Theta(X) \rightarrow \Theta(\Gamma)$ consists of 2^{g-1} odd theta characteristics and 2^{g-1} even theta characteristics.*

Proof. By Corollary 3.2, there exists $\mathcal{L} \in \Theta(X)$ such that $\text{Trop}(\mathcal{L}) = L_{\gamma}$. The preimage of L_{γ} is then precisely $\mathcal{L} + \Lambda$, which has order 2^g . By Corollary 4.4, the map $q_{\mathcal{L}} : \Lambda \rightarrow \mathbb{Z}/2\mathbb{Z}$ is linear. The kernel of this map is precisely those divisors $\mathcal{D} \in \Lambda$ such that $\mathcal{L} + \mathcal{D}$ has the same parity as \mathcal{L} . Thus, if the map is trivial, then all theta characteristics in the preimage of L_{γ} have the same parity as \mathcal{L} , whereas if the map is surjective, then exactly half of the theta characteristics in the preimage of L_{γ} have the same parity as \mathcal{L} .

If \mathcal{L} is odd, however, then $q_{\mathcal{L}}$ has Arf invariant -1, and therefore does not have any isotropic subspaces of dimension g . In particular, the restriction of $q_{\mathcal{L}}$ to Λ cannot be trivial. It follows that the preimage of L_{γ} consists either entirely of even theta characteristics, or half even and half odd. Summing over all γ , we see that the number of odd theta characteristics on X is $(2^g - m)2^{g-1}$, where m is the number of cycles $\gamma \in H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$ such that

the preimage of L_γ contains only even theta characteristics. It is well known, however, that an algebraic curve possesses exactly $(2^g - 1)2^{g-1}$ odd theta characteristics, hence $m = 1$. \square

Example 4.6. Returning to Example 4.1, the 28 odd theta characteristics on a non-hyperelliptic genus 3 curve are the points of intersection of the bitangent lines to the curve in its canonical embedding. By Theorem 4.5, each of the 7 effective theta characteristics on Γ pictured in Figure 1 is the specialization of exactly 4 odd theta characteristics on X . In this example, the 7 effective theta characteristics on Γ are rigid in the sense that they each have only one effective divisor in their class, so in fact the points of intersection of the 28 bitangents specialize 4-to-1 to precisely these *divisors* on Γ , not just these divisor *classes*.

5. DOUBLE COVERS

Recall that there is a one-to-one correspondence between unramified double covers of an algebraic curve X and 2-torsion points in $\text{Jac}(X)$. We now explore the relationship between unramified double covers of a metric graph and 2-torsion points in its Jacobian. We first review the necessary material on augmented metric graphs and harmonic morphisms. For a more detailed discussion of harmonic morphisms, see [ABBR15].

Definition 5.1. *An augmented metric graph is a metric graph Γ , together with a function $g : \Gamma \rightarrow \mathbb{Z}_{\geq 0}$, called the genus function, such that $g(x) = 0$ for all but finitely many $x \in \Gamma$.*

The metric graphs we have been considering thus far can be considered augmented graphs for which the genus function is identically zero. By the Jacobian of an augmented graph (Γ, g) we will mean the Jacobian of the (unaugmented) metric graph $\Gamma^\#$ obtained by adding $g(x)$ loops (of arbitrary length) based at x for each point $x \in \Gamma$. None of our results below depend on the lengths of these loops.

Definition 5.2. *A continuous map $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ of metric graphs is called a morphism if there exist vertex sets $\tilde{V} \subset \tilde{\Gamma}, V \subset \Gamma$, such that $\varphi(\tilde{V}) \subseteq V$, $\varphi^{-1}E(\Gamma) \subseteq E(\tilde{\Gamma})$, and the restriction of φ to any edge \tilde{e} of $\tilde{\Gamma}$ is dilation by some factor $d_{\tilde{e}}(\varphi) \in \mathbb{Z}_{\geq 0}$. A morphism is called finite if $d_{\tilde{e}}(\varphi) > 0$ for all edges \tilde{e} .*

The tropical analogue of a map between algebraic curves is not only a morphism, but a harmonic morphism.

Definition 5.3. *A finite morphism of metric graphs is called harmonic at $\tilde{x} \in \tilde{\Gamma}$ if the sum*

$$d_{\tilde{x}}(\varphi) := \sum_{\tilde{v} \in T_{\tilde{x}}(\tilde{\Gamma}), \varphi(\tilde{v})=v} d_{\tilde{v}}(\varphi)$$

is independent of the choice of tangent vector $v \in T_{\varphi(\tilde{x})}(\Gamma)$. The number $d_{\tilde{x}}(\varphi)$ is called the degree of the harmonic morphism at \tilde{x} . The morphism is called harmonic if it is surjective and harmonic at every point $\tilde{x} \in \tilde{\Gamma}$. In this case, the number $\deg(\varphi) = \sum_{\varphi(\tilde{x})=x} d_{\tilde{x}}(\varphi)$ is independent of x , and is called the degree of the harmonic morphism φ .

Our primary interest here is in unramified harmonic morphisms. Given a point $\tilde{x} \in \tilde{\Gamma}$, the ramification of φ at \tilde{x} is

$$R_{\tilde{x}} = d_{\tilde{x}}(\varphi)(2 - 2g(\varphi(\tilde{x}))) - (2 - 2g(\tilde{x})) - \sum_{\tilde{v} \in T_{\tilde{x}}(\tilde{\Gamma})} (d_{\tilde{v}}(\varphi) - 1),$$

or equivalently, $R_{\tilde{x}} = K_{\tilde{\Gamma}}(\tilde{x}) - \varphi^*(K_{\Gamma})(\tilde{x})$. We say that the harmonic morphism φ is *unramified* if $R_{\tilde{x}} = 0$ for all $\tilde{x} \in \tilde{\Gamma}$.

We often refer to an unramified harmonic morphism of degree 2 as an *unramified double cover*. Throughout this section, we are only interested in such covers in which the target graph Γ is unaugmented. However, we cannot rule out the possibility that the genus function of the covering graph $\tilde{\Gamma}$ is non-trivial. Given an unramified double cover $\varphi : \tilde{\Gamma} \rightarrow \Gamma$, we define the *dilation cycle* $\gamma(\varphi)$ to be the set of points of Γ in the image of a dilated edge. More precisely,

$$\gamma(\varphi) := \{x \in \Gamma \mid \exists \tilde{x} \in \tilde{\Gamma}, \tilde{v} \in T_{\tilde{x}}(\tilde{\Gamma}) \text{ with } \varphi(\tilde{x}) = x \text{ and } d_{\tilde{v}}(\varphi) = 2\}.$$

We now describe the genus function of an unramified double cover.

Lemma 5.4. *Let $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ be an unramified harmonic morphism of degree 2, with Γ unaugmented.*

- (1) *If $x \notin \gamma(\varphi)$, then $\varphi^{-1}(x)$ consists of two points of genus 0.*
- (2) *If $x \in \gamma(\varphi)$, then $\varphi^{-1}(x)$ is a single point of genus $\frac{1}{2} \deg_{\gamma(\varphi)}(x) - 1$, where $\deg_{\gamma(\varphi)}(x)$ is the number of tangent vectors at x contained in $\gamma(\varphi)$.*

Proof. Since φ is unramified and the genus function of Γ is trivial, for any point $\tilde{x} \in \tilde{\Gamma}$ the ramification formula gives us

$$2g(\tilde{x}) - 2 = \sum_{\tilde{v} \in T_{\tilde{x}}(\tilde{\Gamma})} (d_{\tilde{v}}(\varphi) - 1).$$

As φ is finite and degree 2, each dilation factor $d_v(\varphi)$ is either 1 or 2. It follows that the right hand side of the expression above is equal to $\deg_{\gamma(\varphi)}(x)$. \square

The following corollary explains our choice of terminology.

Corollary 5.5. *The dilation cycle $\gamma(\varphi)$ is a cycle in Γ .*

Proof. By Lemma 5.4, for every point $x \in \Gamma$ there is an even number of tangent vectors in $T_x(\Gamma)$ that are contained in $\gamma(\varphi)$. It follows that $\gamma(\varphi)$ is a cycle. \square

Remark 5.6. In many applications, one is primarily interested in trivalent metric graphs, since these form an open dense subset of the moduli space of tropical curves. By Lemma 5.4, we see that if Γ is trivalent, then the genus function on $\tilde{\Gamma}$ is identically zero. In this case, there is no ambiguity with how one defines the Jacobian of the cover $\tilde{\Gamma}$.

Our first result on double covers concerns the kernel of the pullback map on divisors.

Proposition 5.7. *Let $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ an unramified harmonic morphism of degree 2 with Γ unaugmented. Then*

$$\ker \varphi^* = \{D_\gamma \mid \gamma \subseteq \gamma(\varphi)\},$$

where D_γ is the divisor associated to the cycle γ as defined in (4.1).

Proof. First, note that the composition $\varphi_*\varphi^*$ is simply multiplication by 2, so the kernel of φ^* must be contained in the set of 2-torsion points.

Recall the definition of the piecewise linear functions d_v and d_γ from the beginning of §4. We show that for every cycle γ , the divisor φ^*D_γ is principal if and only if the piecewise linear function $\varphi^*(d_\gamma - d_v)$ has even slope at every tangent vector in $\tilde{\Gamma}$. Indeed, given $f \in \text{PL}(\tilde{\Gamma})$ with $\text{div}(f) = \varphi^*D_\gamma$, we see that $\text{div}(2f) = \text{div}(\varphi^*(d_\gamma - d_v))$. It follows that $2f - (d_\gamma - d_v)$ must be a constant. In other words, up to translation by a constant, we have

$$f = \frac{1}{2}\varphi^*(d_\gamma - d_v),$$

which is well-defined if and only if all of the slopes of $\varphi^*(d_\gamma - d_v)$ are even.

Now, note that by definition $d_\gamma - d_v$ has odd slope at every tangent vector in γ , and even slope at every tangent vector in $\Gamma \setminus \gamma$. Since the slope of $\varphi^*(d_\gamma - d_v)$ at a tangent vector v in $\tilde{\Gamma}$ is precisely $d_v(\varphi)$ times the slope of $d_\gamma - d_v$ at $\varphi(v)$, we see that $\varphi^*(d_\gamma - d_v)$ has odd slopes at tangent vectors in $\varphi^{-1}(\gamma \setminus \gamma(\varphi))$. We conclude that the slopes of $\varphi^*(d_\gamma - d_v)$ are everywhere even (and therefore D_γ is principal) precisely when $\gamma \subseteq \gamma(\varphi)$. \square

Note in particular that there exist double covers $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ for which φ^* is injective. These are precisely the covers for which there is no dilation, that is $\gamma(\varphi) = 0$. In other words, these are the covering spaces of Γ . A graph of genus g admits exactly 2^g covering spaces of degree 2, which can be described as follows. Choose a spanning tree T of G . Any degree 2 covering space will contain a disjoint union $T \sqcup T'$ of two copies of T . Now, for each of the g edges in the complement of T , there are 2 possible ways to lift it to the double cover. If the edge connects vertices v_1 to v_2 , we may lift it to two edges connecting v_1 to v_2 and v'_1 to v'_2 , or to two edges connecting v_1 to v'_2 and v'_1 to v_2 . To see that this is the complete set of degree 2 covering spaces of Γ , note that the fundamental group of Γ is invariant under homotopy, and thus such covering spaces are in bijection with degree 2 covering spaces of the rose obtained by contracting T . An exposition of this can be found in [Wal76].

For unramified harmonic double covers we have the following.

Lemma 5.8. *Let γ be a cycle in an unaugmented metric graph Γ , and let h be the sum of the genera of the connected components of $\Gamma \setminus \gamma$. Then up to isomorphism, there are 2^h unramified harmonic morphisms $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ of degree 2 with dilation cycle γ .*

Proof. This follows immediately from the fact that the preimage of each connected component of $\Gamma \setminus \gamma$ is a covering space. \square

We now consider the relationship between unramified double covers of the graph Γ and the curve X . Any unramified double cover $\tilde{\varphi} : \tilde{X} \rightarrow X$ descends to an unramified double cover of skeletons. The following proposition establishes the converse.

Lemma 5.9. *Let γ be a cycle in an unaugmented metric graph Γ and let h be the sum of the genera of the connected components of $\Gamma \setminus \gamma$. Let $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ be a degree two unramified harmonic morphism with dilation cycle γ . Then there are 2^{g-h} unramified double covers $\tilde{\varphi} : \tilde{X} \rightarrow X$ that specialize to φ .*

Proof. Let m be the number of connected components of $\Gamma \setminus \gamma$. If V is the vertex set consisting of all points of Γ of valence different from 2, we see that the number of edges of γ in the corresponding model for Γ is $g+m-h-1$. The cover $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ has 2^m automorphisms, given by involution on the preimages of the connected components of $\Gamma \setminus \gamma$. Hence, by [CMR16, Theorem 2], there are $2^{\frac{2^{g+m-h-1}}{2^m}} = 2^{g-h}$ double covers $\tilde{\varphi} : \tilde{X} \rightarrow X$ that specialize to φ . \square

An immediate consequence of Lemmas 5.8 and 5.9 is the following.

Proposition 5.10. *Let γ be a cycle in an unaugmented metric graph Γ . Then there are 2^g unramified double covers $\tilde{\varphi} : \tilde{X} \rightarrow X$ specializing to harmonic morphisms of degree two with dilation cycle γ .*

Proof. By Lemma 5.8, there are 2^h unramified harmonic morphisms of degree 2 with dilation cycle γ . By Lemma 5.9, each of these lifts to 2^{g-h} double covers of X . Hence there are $2^h \cdot 2^{g-h} = 2^g$ such double covers of X . \square

We now describe the connection between covers of Γ and covers of X .

Theorem 5.11. *Let \mathcal{D} be a 2-torsion point in $\text{Jac}(X)$. Let $\tilde{\varphi} : \tilde{X} \rightarrow X$ be the corresponding double cover and $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ the specialization of this cover. Then $\text{Trop}(\mathcal{D}) = D_{\gamma(\varphi)}$.*

Proof. Consider the following diagram.

$$\begin{array}{ccc} \text{Jac}(X) & \xrightarrow{\text{Trop}} & \text{Jac}(\Gamma) \\ \downarrow \tilde{\varphi}^* & & \downarrow \varphi^* \\ \text{Jac}(\tilde{X}) & \xrightarrow{\text{Trop}} & \text{Jac}(\tilde{\Gamma}) \end{array}$$

Since $\bar{\varphi}^*\mathcal{D} = 0$, we see that $\varphi^* \text{Trop}(\mathcal{D}) = 0$ as well. By Proposition 5.7, it follows that $\text{Trop}(\mathcal{D}) = D_\gamma$ for some $\gamma \subseteq \gamma(\varphi)$.

Suppose by induction that the theorem is true for every cycle properly contained in $\gamma(\varphi)$. That is, for any cycle $\gamma' \subsetneq \gamma(\varphi)$, every double cover $\bar{\psi}$ of X that specializes to a harmonic morphism ψ of Γ with dilation cycle γ' corresponds to a divisor $D_{\bar{\psi}}$ on X that specializes to $D_{\gamma'}$. For any such γ' , there are 2^g divisor classes specializing to $D_{\gamma'}$ (since this is the order of the kernel Λ), and by Proposition 5.10 there are 2^g double covers of X specializing to harmonic morphisms of Γ with dilation cycle γ' . Because these two sets have the same size, every divisor specializing to $D_{\gamma'}$ corresponds to a double cover that specializes to a harmonic morphism with dilation cycle γ' . It follows that the containment $\gamma \subseteq \gamma(\varphi)$ cannot be strict. \square

Remark 5.12. It follows that the combinatorics of the graph Γ “sees” the entire 2-torsion subgroup $\text{Jac}_2(X)$: By the preceding theorem, there is a canonical bijection between the kernel Λ and the set of degree 2 covering spaces of Γ . Therefore, $\text{Jac}_2(X)$ is an extension of $\text{Jac}_2(\Gamma) \cong H_1(\Gamma, \mathbb{Z}/2\mathbb{Z})$ by the set of covering spaces. In fact, this extension splits (non-canonically) as all the groups involved are vector spaces.

Example 5.13. As an example, we consider the double cover of our earlier example Γ by the cube $\tilde{\Gamma}$, again with all edge lengths 1, pictured in Figure 2. By Lemma 5.9, there exists a unique double cover $\bar{\varphi} : \tilde{X} \rightarrow X$ specializing to this double cover of Γ .

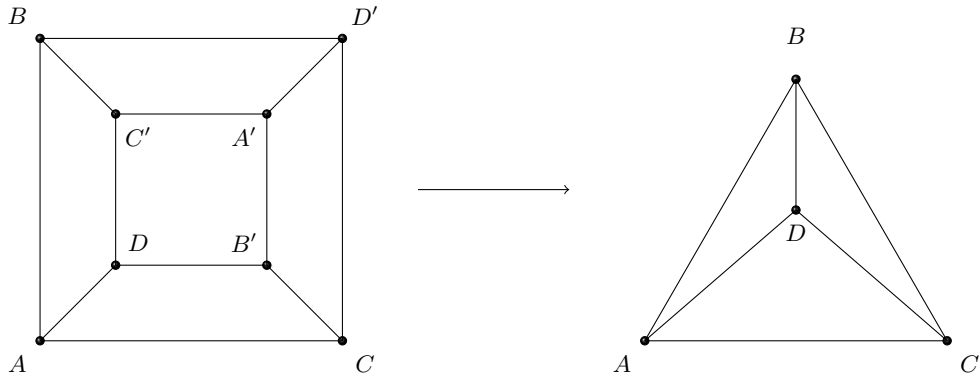


FIGURE 2. A double cover of Γ by a cube

Figure 3 depicts the pullback of D_γ for each of the two effective theta characteristics L_γ pictured in Figure 1. The divisor on the left corresponds to the case where γ is a triangle, whereas the divisor on the right corresponds to the square. We leave it to the reader to verify that neither of these divisors is equivalent to zero.

Example 5.14. By Lemma 5.8, for any cycle $\gamma \neq 0$ in the graph Γ of the previous example, there is a unique double cover $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ with dilation

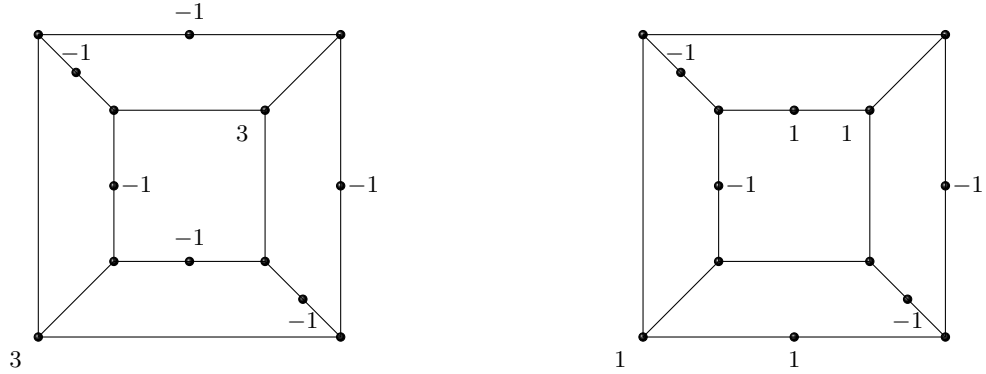


FIGURE 3. Pullbacks of two 2-torsion divisors from the complete graph to the cube

cycle γ . These are depicted in Figures 4 and 5, with the dilation cycle marked by a solid line. By Lemma 5.9, each of these lifts to 8 distinct unramified double covers of the curve X .

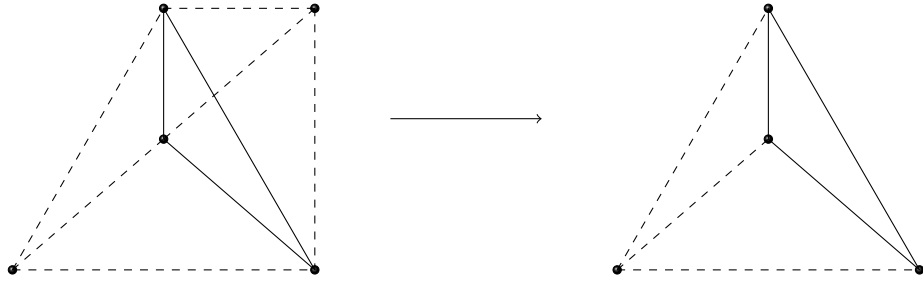


FIGURE 4. The unique double cover with dilation cycle a given triangle in Γ

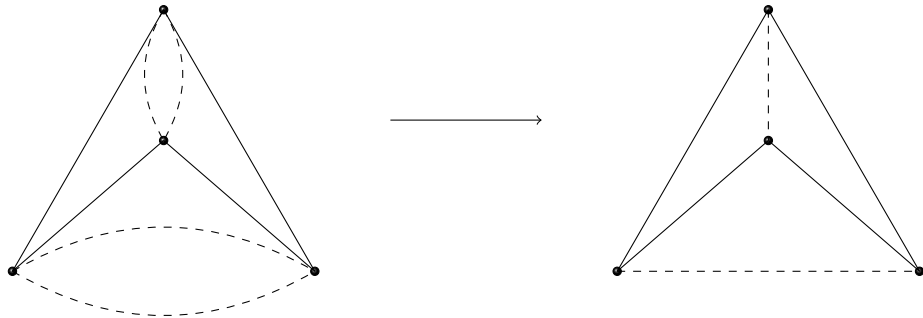


FIGURE 5. The unique double cover with dilation cycle a given square in Γ

6. PRYM VARIETIES

We now study the kernel of the pushforward map on divisors. Recall that the Jacobian of an augmented graph Γ is, by definition, the Jacobian of the unaugmented graph $\Gamma^\#$ obtained by attaching $g(v)$ loops of arbitrary length at every point v . Those loops are referred to as *virtual loops*.

Given a degree 2 unramified harmonic morphism $\varphi : \tilde{\Gamma} \rightarrow \Gamma$, consider the involution $\iota : \tilde{\Gamma}^\# \rightarrow \tilde{\Gamma}^\#$ that sends a point x of $\tilde{\Gamma}$ to the other preimage point of $\varphi(x)$, and sends a point on a virtual loop to the other point equidistant from the base of the same loop. Extending by linearity, ι defines an involution $\iota : \text{Pic}(\tilde{\Gamma}) \rightarrow \text{Pic}(\tilde{\Gamma})$.

Proposition 6.1. *Let $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ be a degree 2 unramified harmonic morphism. If $\gamma(\varphi) = 0$, then the kernel of the pushforward map*

$$\varphi_* : \text{Jac}(\tilde{\Gamma}) \rightarrow \text{Jac}(\Gamma)$$

has two connected components, namely

$$\ker \varphi_* = (\text{Id} - \iota) \text{Pic}^0(\tilde{\Gamma}) \cup (\text{Id} + \iota) \text{Pic}^1(\tilde{\Gamma}).$$

If $\gamma(\varphi) \neq 0$, it has only one connected component.

Proof. Consider the natural pullback map on harmonic 1-forms (which we denote ψ^* to avoid confusion with φ^*)

$$\psi^* : \Omega(\Gamma) \rightarrow \Omega(\tilde{\Gamma}).$$

It is easy to see that this map is injective. Dualizing, we obtain a surjective map between the universal covers of the Jacobians

$$\psi_* : \Omega(\tilde{\Gamma})^* \rightarrow \Omega(\Gamma)^*.$$

We write $\Omega(\tilde{\Gamma}/\Gamma)^*$ for the kernel of ψ_* . The map ψ_* descends to the map φ_* of Jacobians by quotienting out $H_1(\Gamma, \mathbb{Z})$ and $H_1(\tilde{\Gamma}, \mathbb{Z})$, respectively. The kernel of φ_* therefore corresponds to $\psi_*^{-1}H_1(\Gamma, \mathbb{Z})$. Since the map $\psi_*\psi^*$ is multiplication by 2, it follows that

$$\psi_*^{-1}H_1(\Gamma, \mathbb{Z}) = \Omega(\tilde{\Gamma}/\Gamma)^* + \frac{1}{2}\psi^*H_1(\Gamma, \mathbb{Z}).$$

Passing to the quotient, we see that $\ker \varphi_*$ consists of translates of the $(g-1)$ dimensional torus

$$P(\varphi) := \Omega(\tilde{\Gamma}/\Gamma)^*/(H_1(\tilde{\Gamma}, \mathbb{Z}) \cap \Omega(\tilde{\Gamma}/\Gamma)^*) \subset \text{Jac}(\tilde{\Gamma})$$

by elements of $\varphi^* \text{Jac}_2(\Gamma)$.

To compute the number of connected components, it therefore suffices to determine when the pullbacks of $D, E \in \text{Jac}_2(\Gamma)$ lie in the same component. Note that

$$\varphi^* \text{Jac}_2(\Gamma) \subset \text{im}(\text{Id} - \iota) \subseteq \ker \varphi_*,$$

so if D and E lie in the same connected component of $\text{im}(\text{Id} - \iota)$, then they lie in the same connected component of $\ker \varphi_*$.

Let D be a divisor of degree k on $\tilde{\Gamma}$. Choose divisors D_1, D_2 such that $D = D_1 + D_2$, $\deg(D_1) = \lceil \frac{k}{2} \rceil$, $\deg(D_2) = \lfloor \frac{k}{2} \rfloor$. Then

$$D - \iota(D) = (D_1 - \iota(D_2)) - \iota(D_1 - \iota(D_2)).$$

Since $D_1 - \iota(D_2)$ has degree either 0 or 1, we see that

$$\text{im}(\text{Id} - \iota) = (\text{Id} - \iota) \text{Pic}^0(\tilde{\Gamma}) \cup (\text{Id} - \iota) \text{Pic}^1(\tilde{\Gamma}).$$

It follows that $\ker \varphi_*$ has at most 2 connected components.

If $\gamma(\varphi) \neq 0$, then any point $x \in \varphi^{-1}(\gamma(\varphi))$ is fixed by the involution ι . Then, for any divisor $D \in \text{Pic}^0(\tilde{\Gamma})$, we have

$$D - \iota(D) = (D + x) - \iota(D + x),$$

so

$$(\text{Id} - \iota) \text{Pic}^0(\tilde{\Gamma}) = (\text{Id} - \iota) \text{Pic}^1(\tilde{\Gamma}),$$

hence $\ker \varphi_*$ has only one connected component.

On the other hand, if $\gamma(\varphi) = 0$, then by Proposition 5.7, φ^* is injective, so $|\varphi^* \text{Jac}_2(\Gamma)| = 2^g$. Since the $(g-1)$ -dimensional torus $P(\varphi)$ contains only 2^{g-1} elements of order 2, we see that $\ker \varphi_*$ must contain at least two connected components. □

Definition 6.2. *We define the Prym variety $P(\varphi)$ to be the connected component of $\ker \varphi_*$ containing 0. That is,*

$$P(\varphi) := (\text{Id} - \iota) \text{Pic}^0(\tilde{\Gamma}).$$

Note that the Prym variety $P(\varphi)$ is a real torus of dimension $g-1$.

On an algebraic curve, there is an alternative way to define the Weil pairing using the algebraic version of the construction above. Given a 2-torsion point \mathcal{D} in the Jacobian of an algebraic curve X , let $\bar{\varphi} : \tilde{X} \rightarrow X$ be the corresponding unramified double cover. Since $\bar{\varphi}_* \bar{\varphi}^*$ is multiplication by 2, the pullback of any 2-torsion point $\mathcal{E} \in \text{Jac}_2(X)$ is in the kernel of $\bar{\varphi}_*$. One can show that the Weil pairing $\lambda(\mathcal{D}, \mathcal{E})$ is 0 if $\bar{\varphi}^* \mathcal{E}$ is contained in the Prym variety, and 1 if it is contained in the other connected component. In this way, the Weil pairing is best understood as a pairing between $\text{Jac}_2(X)$ and the set of unramified double covers of X .

In the tropical setting, this construction only makes sense when the double cover has dilation cycle $\gamma(\varphi) = 0$. For such covers, however, the specialization of the Weil pairing is well-behaved.

Proposition 6.3. *Let $\mathcal{D} \in \Lambda$, let $\bar{\varphi} : \tilde{X} \rightarrow X$ be the corresponding double cover, and $\varphi : \tilde{\Gamma} \rightarrow \Gamma$ the associated double cover of skeletons. Then, for any $\mathcal{E} \in \text{Jac}_2(X)$, we have $\lambda(\mathcal{D}, \mathcal{E}) = 0$ if and only if*

$$\varphi^* \text{Trop}(\mathcal{E}) \in P(\varphi).$$

Proof. Since Λ is isotropic for the Weil pairing, the linear map $\lambda(\mathcal{D}, \cdot)$ factors through $\text{Jac}_2(\Gamma)$. Exactly half the elements of $\text{Jac}_2(\Gamma)$ map to 0 and half to 1. Similarly, the $(g - 1)$ dimensional torus $P(\varphi)$ contains 2^{g-1} 2-torsion points. Since φ^* is injective, half the elements of $\varphi^* \text{Jac}_2(\Gamma)$ are contained in $P(\varphi)$ and half are contained in the other connected component. By pigeonhole it therefore suffices to show that

$$\text{Trop}((\text{Id} - \iota) \text{Jac}(\tilde{X})) \subseteq (\text{Id} - \iota) \text{Jac}(\tilde{\Gamma}).$$

This follows from the commutativity of the following square.

$$\begin{array}{ccc} \text{Jac}(\tilde{X}) & \xrightarrow{\text{Trop}} & \text{Jac}(\tilde{\Gamma}) \\ \downarrow \iota & & \downarrow \iota \\ \text{Jac}(\tilde{X}) & \xrightarrow{\text{Trop}} & \text{Jac}(\tilde{\Gamma}) \end{array}$$

□

Example 6.4. We now return to the example of the double cover of the complete graph Γ by the cube $\tilde{\Gamma}$. By Proposition 5.10, there exists a unique double cover $\tilde{\varphi} : \tilde{X} \rightarrow X$ specializing to this double cover of Γ . We let $\mathcal{D} \in \text{Jac}_2(X)$ be the 2-torsion divisor corresponding to the double cover $\tilde{\varphi}$.

The divisor depicted in Figure 6 is equivalent to the divisor pictured on the left in Figure 3. This divisor is easily seen to be in $(\text{Id} - \iota) \text{Pic}^1(\tilde{\Gamma})$. In other words, it is not contained in the Prym variety $P(\varphi)$. Conversely, the divisor depicted in Figure 7 is equivalent to the divisor on the right in Figure 3. This divisor is easily seen to be in $(\text{Id} - \iota) \text{Pic}^0(\tilde{\Gamma})$. In other words, it is contained in the Prym variety $P(\varphi)$. By Proposition 6.3, therefore, if $\mathcal{E} \in \text{Jac}_2(X)$ specializes to D_γ , then $\lambda(\mathcal{D}, \mathcal{E}) = 1$ when γ is a triangle, and $\lambda(\mathcal{D}, \mathcal{E}) = 0$ when γ is a square.

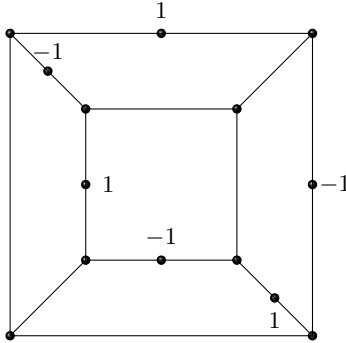


FIGURE 6. A divisor equivalent to the divisor on the left in Figure 3

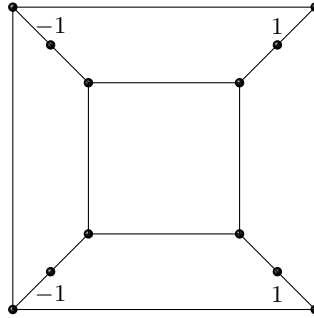


FIGURE 7. A divisor equivalent to the divisor on the right in Figure 3

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