AP Calculus BC Project
Arc Length

In computing the length of a curve we are often unable to apply the Fundamental Theorem of Calculus because the antiderivatives that arise are not expressible in terms of elementary functions. Recall that the length of a curve given by a function $y = f(x)$ from $x = a$ to $x = b$ is given by

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$

In this lab we are going to look at two examples of arclength and one of surface area.

How to Break a Guitar String

There are two models for the shape of a guitar string. The first assumes that the string is plucked and the shape is like a tent and the second assumes that the string has been excited by a vibration and is in the shape of a sine curve. All you air-guitarists will know that a string will vibrate without touching it if another string tuned to the same notes is plucked.

Let's assume our guitar string is 3 feet long and the shape of the string when plucked 6 inches from the end is given by

$$f(x) = \begin{cases} 2ax & 0 \le x \le 0.5 \\ -\dfrac{2a}{5}x + \dfrac{6a}{5} & 0.5 \le x \le 3 \end{cases}$$

where a is the distance above the rest position the string is pulled and this is given in feet (not inches).

For the second model the shape of a 3 foot guitar string vibrating in the fundamental frequency, or first mode, has the shape

$$f(x) = a\sin\left(\frac{\pi}{3}x\right).$$

For the second model a is called the *amplitude* and x and y are measured in feet.

You want to enter these two models into your calculator as Y1 and Y4. You need to recall how to enter piecewise defined functions into your calculator. Leave a in the definition of the two functions and we will assign it a value a little later.

Problem 1: *For the first model graph the guitar string for a taking the three values* $a = -0.2$, $a = 0$, *and* $a = 0.4$ *for* $0 \le x \le 3$ *and* $-0.4 \le y \le 0.4$ *all on the same graph as Y1, Y2, and Y3. For each graph, what does a represent? Do the same for the second model, putting the three models in Y4, Y5, and Y6.*

In order to get started with this let me give you a hint on how to do one of the plots.

Problem 2: *For the first model find the length of the string for each of the a values in Problem 1. Also, do this for the second model.*

You can get an exact answer for each of the cases using the first model. If you cannot get an exact answer in the second model, use Simpson's Rule to get an approximation. You have the program RIEMANN which can get you to a good Simpson approximation.

Problem 3: *Suppose that it has been determined experimentally that the string breaks if it is stretched to a length of 3.1 feet. For each of the first and second model determine the amplitude a that will break the string. First, try finding the length of the string for a general a exactly using the definite integral, then set up an appropriate equation and solve it. If you cannot find the length of the string for a general a then use trial and error with different, well chosen, a values until you find the one that gives a length of 3.1 feet accurate to two decimal places. Sketch the string for the a value which breaks the string for both models on one set of axes.*

Surface Area and Rugby Balls

A rugby ball looks like a football with rounder ends. In fact a rugby ball is a prolate spheroid. Take the upper half of an ellipse and rotate it about the x-axis and you get a rugby ball. To make a rugby ball that is 16 inches long and 10 inches in diameter you need 4 pieces of a leatheroid material which are sewn together to make the ball. The defining ellipse is

$$25x^2 + 64y^2 = 1600.$$

To sew the pieces of leather together it must be done along two seams. Each of these seams a copy of the ellipse. The amount of thread required to sew the seams is 60% **more** than the length of the seam.

Problem 4: *How much thread is required to sew the seams on the rugby ball?*

Now, we need to know how much leatheroid material it takes to cover the rugby ball. This means we need to know the surface area of the solid of revolution. Since we are rotating about the x-axis the surface area is given by

$$SA = \int_a^b 2\pi y\, ds = \int_a^b 2\pi f(x)\sqrt{1+[f'(x)]^2}\, dx.$$

Problem 5: *How much material does it take to cover the rugby ball? Assuming a cylindrical pig, 5 feet long and 24 inches in diameter, how many rugby balls will this* pigskin *make?*