AP Calculus BC Project
The Dam Problem:
Constructing a Dam across the River

You are an engineer preparing to oversee construction of a dam across the river. The dam will be made of concrete designed as in the following diagram.

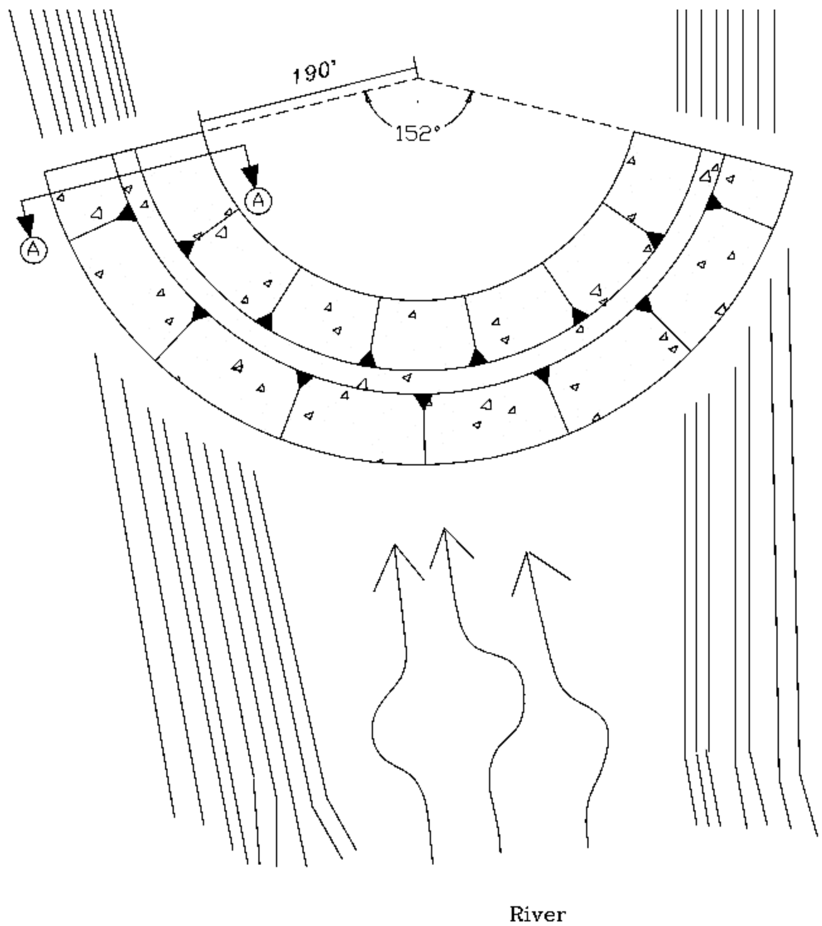

Figure 1: Top view of the dam

In most dams the cross-sectional area, a side view of which is shown in Figure 2, varies across the length of the dam. For those of us just learning to “walk,” we will hold the cross-sectional area constant — a simplification of a real world problem.

The two ends of the dam that join the rock cliff must be prepared so that a strong connection is made between the rock cliff and the dam itself. Your first assignment is to determine the total area of the two ends of the dam.

I. Finding the Cross-sectional Area

From the sketch of the cross-sectional area in Figure 2 you can break the problem into three parts: Part I is the area under the curve $y = 0.0225x^2 + 2.3x$, $0 \le x \le 60$. Part II is the area of a rectangle 438 feet high by 20 feet wide. Finally, Part III is the area under the line $y = -7.3x + 1022$, $80 \le x \le 140$.

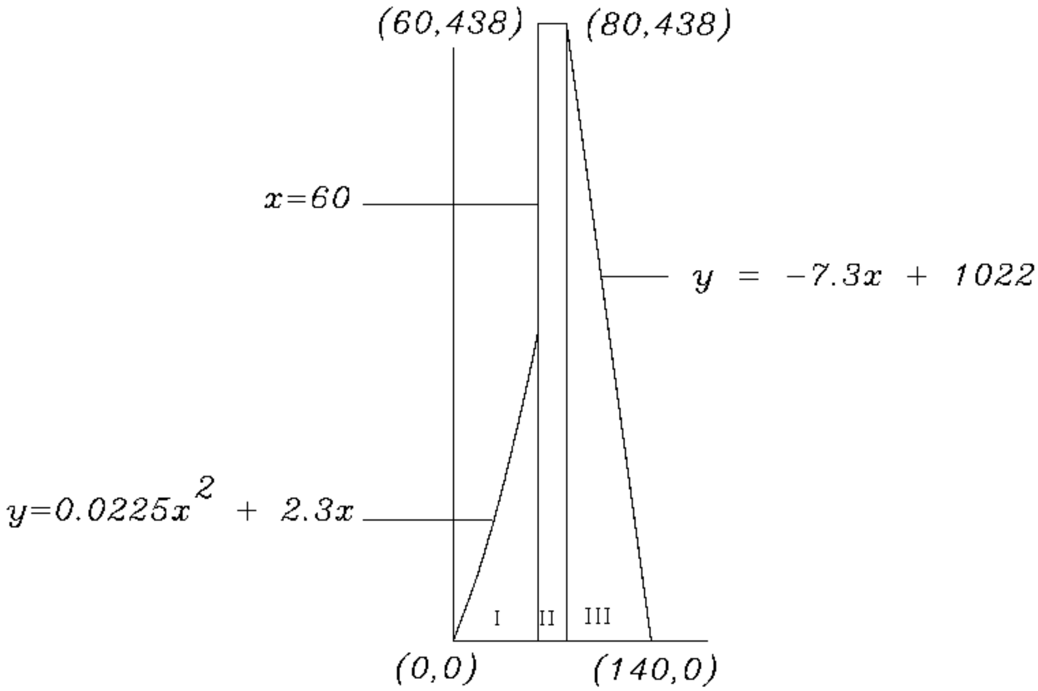

Figure 2: Side view of dam

The sum of the areas of the three parts gives the area of one end. Since the two ends are the same, we only need to double this value to find the total area of the two ends of the dam.

Each of these areas is simple to compute. We will compute the areas of Parts I and III *via* integration, and the area of Part II by geometric formula.

$$\text{Area(Part I)} = \int_0^{60} (0.0225x^2 + 2.3x)\,dx =$$

$$\text{Area(Part II)} = \text{width} \times \text{height} =$$

$$\text{Area(Part III)} = \int_{80}^{140} (-7.3x + 1022)\,dx =$$

The area of the ends is then the sum of these three areas.

Problem 1: *What is this area? What is this in square yards?*

II. Volume of Construction

How many cubic feet of concrete are needed to construct the dam?

The total volume may be determined by revolving the cross-sectional area through 152° about the y-axis at a radius of 190 feet. Since the rotation point is 190 feet away from the nearest face of the dam, we must translate the equations a distance of x = 190 feet from the origin of the cross-section so that the rotation takes place about the y-axis. Then integrate each of the three parts (Parts I, II, and III) as volumes of revolution.

How do you translate a function by 190 feet? It is quite simple. All that we need to do is to substitute $(x-190)$ for x in each of the functions defining the cross-section of the dam.

So for each of Parts I and III above we need you to substitute $(x-190)$ for x and you will get TransPartI and TransPartIII. Why don't we do the same for Part II?

If you want, you can expand the polynomials in TransPartI and TransPartII. It is not absolutely necessary. The cross-section of the dam can be described with these new translated functions, as shown in Figure 3.

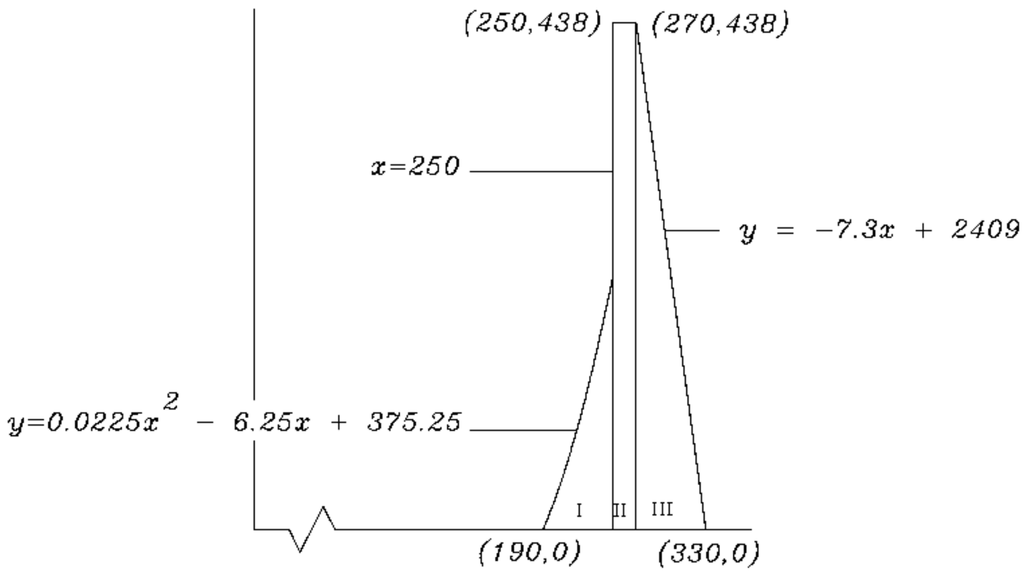

Figure 3: Side view of the dam

Now, you have to know how to do an integral of rotation about the y-axis. As we have covered if the portion of the graph of $f(x)$ for $a \le x \le b$ is rotated around the y-axis, the volume of the resulting solid is given by

$$\text{Volume} = \int_a^b 2\pi x \cdot f(x)\,dx$$

Problem 2: *Find the volumes of the solids given by rotating each of the three parts around the y-axis. Be extra careful of the limits of integration. They have changed since we made a change of variables with the x-values.*

The sum of these three calculations produces a volume of a solid formed by a complete 360-degree revolution about the y-axis. Since the dam only fills 152 degrees, the final step in calculating the total volume of the dam is to multiply the sum by a constant. We only need 152/360 of the complete circle.

Problem 3: *What is the constant? It is a ratio. What is the total volume of concrete to be used in the dam? What is this volume in cubic yards? You construction firm will place an order for how many yards of concrete?*}

Spraying the Surface with a Curing Compound

There is more to constructing the dam than just pouring concrete. The outside surfaces of the dam must be given some special treatment. The front and rear faces of the dam must be covered with a curing compound. You have to determine the amount of compound needed; assuming that one gallon of curing compound covers 40 square feet of surface area.

First, you have to determine the surface area formed by revolving the functions for TransPartI and TransPartIII, the front and rear faces, around the y-axis. If the function $f(x)$ has a derivative for $a \le x \le b$ and the portion of the curve $y = f(x)$ for $a \le x \le b$ is rotated around the y-axis, the surface area of the surface generated is given by

$$\text{Surface Area} = \int_a^b 2\pi x\sqrt{1+[f'(x)]^2}\,dx.$$

Since the front of the dam comes in one piece, the surface area of the front

of the dam is

$$\text{Surface Area}_{Front} = \int_{270}^{330} 2\pi x\sqrt{1+\left[\frac{d}{dx}TransPartIII\right]^2}\,dx.$$

Problem 4: *How can you calculate the area of the rear surface? It is made up of two parts, the curve defined by the expression TransPartI and the vertical portion that extends the height of the dam from 219 to 438 feet. Calculate the surface area of each piece and sum them together.*

Problem 5: *What is the total surface area and how many gallons of compound have to be ordered?*

Problem 6: *How much steel is needed to cover the top edge of the dam?*

Problem 7: *The base of the dam will be embedded 5 feet deep into the rock river bed. How much rock must be excavated to build the dam's foundation?*

Recreational Area

When the dam is completed it will create a 10 mile long lake with an average depth of 45 feet and has a width w in miles given y the equation

$$w = \frac{-12d^2 + 122.2d + 393.52}{5280}.$$

Here d is the distance along the centerline of the lake measured in miles.

Problem 8: *What area does the lake cover? Give your answer in square miles and acres.*