AP Calculus BC Project
Sunshine Power and Energy

In this lab we will study the intensity of the sunshine and the amount of solar energy at a particular point on the surface of the earth and the length of daylight.

Energy and Power

Energy appears in various forms and can often be converted from one form into another. For instance, a solar cell converts the energy in light into electrical energy; a fusion reactor, in changing atomic structures, transforms nuclear energy into heat energy.

Energy is an *extensive* quantity. This means that, for instance, the longer a generator runs, the more electrical energy it produces; the longer a light bulb burns, the more energy it consumes. The rate with respect to time at which some form of energy is produced or consumed is called the *power* output or input of the energy conversion device. Power is an *instantaneous* or *intensive* quantity. By the Fundamental Theorem of Calculus we can compute the total energy transformed between times a and b by integrating the power from a to b.

Power is the rate of change of energy with respect to time: $$P = \frac{dE}{dt}.$$ The total energy over a time period is the integral of power with respect to time: $$E = \int_a^b P\,dt.$$

A common unit of measurement for power is the *watt*, which equals 1 joule per second. One *horsepower* is equal to 746 watts. The *kilowatt-hour* is a unit of energy equal to the energy obtained by using 1000 watts for 1 hour---that is, 3,600,000 joules.

Integrating Sunshine

If we have a horizontal square meter of surface, then the rate at which solar energy is received by this surface — that is, the *intensity* of the solar radiation — is proportional to the sine of the angle A of elevation of the sun above the horizon. Thus, the intensity is highest when the sun is directly overhead ($A = \pi / 2$) and reduces to zero at sunrise and sunset.

Thus, the total energy received on day T must be the product of a constant (which can be determined only by experiment, and which we will ignore) and the integral

$$E = \int_{t_0(T)}^{t_1(T)} \sin(A)\,dt,$$

where t is the time of day measured in hours from noon and $t_0(T)$ and $t_1(T)$ are the times of sunrise and sunset on day T. Note that when the sun is below the horizon, although $\sin(A)$ is negative, the solar intensity is simply zero.

The Sunshine Formula

We will have to work with the following variables:

A the angle of elevation of the sun above the horizon;
l the latitude of a place on the earth's surface;
α inclination of the earth's axis (23.5° or 0.41 radians);
T time of year, measured in days from the first day of summer in the northern hemisphere (June 21);
t time of day, measured in hours from noon.[1]

The formula for $\sin(A)$ is:

$$\sin(A) = \cos(l)\sqrt{1-\sin^2(\alpha)\cos^2\left(\frac{2\pi T}{365}\right)}\cos\left(\frac{2\pi t}{24}\right) + \sin(l)\sin(\alpha)\cos\left(\frac{2\pi T}{365}\right).$$

Applications

At the time of sunset, call it S, we have $A = 0$.[2] Thus,

$$\cos\left(\frac{2\pi S}{24}\right) = -\tan(l)\frac{\sin(\alpha)\cos\left(\frac{2\pi T}{365}\right)}{\sqrt{1-\sin^2(\alpha)\cos^2(\frac{2\pi T}{365})}}.$$

Solve this equation for S. There are two solutions, but S>0 since sunset occurs after noon. Then we get our *sunset formula*

$$S = \frac{12}{\pi}\arccos\left[-\tan(l)\frac{\sin(\alpha)\cos\left(\frac{2\pi T}{365}\right)}{\sqrt{1-\sin^2(\alpha)\cos^2\left(\frac{2\pi T}{365}\right)}}\right].$$

Now, to apply this let's compute the time the sun sets on July 1 here in

---

[1] By noon we mean the moment at which the sun is highest in the sky. To find the correct *noon* find the times of sunrise and sunset and find the time halfway between these two times. It may not be 12:00 but should change very slowly from day to day, ignoring changes to daylight savings time.

[2] If $\pi/2-\alpha <|l|< \pi/2$ (inside the polar circles), there will be some values of t for which the right-hand side of the formula for sin(A) does not lie in the interval [-1,1]. On the days corresponding to these values of t, the sun will never set (*midnight sun*). If $l = \pm\pi/2$, then $\tan(l) = \infty$, and the right-hand side does not make sense at all. This reflects the fact that, at the poles, it is either light all day or dark all day, depending upon the season.

Charlotte---latitude 35°15'. Thus, l=35.25°, $\alpha = 23.5°$, and T=11. Substitute these values into the sunset formula and we get

$$Sunset = \frac{12}{\pi}\arccos\left(-\frac{\tan\left(\frac{35.25\pi}{180}\right)\sin\left(\frac{23.5\pi}{180}\right)\cos\left(\frac{22\pi}{365}\right)}{\sqrt{1-\left(\sin\left(\frac{23.5\pi}{180}\right)\cos\left(\frac{22\pi}{365}\right)\right)^2}}\right)$$

You should get 7.167 hours after *noon*, or 7:10:01 if noon is at 12:00.

Exercise 1: *According to the* Charlotte Observer *sunrise and sunset for September 20 were:*

Sunrise	7:10 am
Sunset	7:24 pm

1. *Find the actual* noon *time in Charlotte on September 20.*
2. *Find the sunset time from the sunset formula. How does this time compare to the time given above?*

Now for a fixed point here on the surface of the earth, sunset time S may be considered as a function of T. When is S the greatest and the least? We only need to differentiate S.

Exercise 2: *Find* $\frac{dS}{dT}$.

The critical points of S occur when $2\pi T / 365 = 0, \pi,$ or 2π. Thus, $T = 0$, $T = 365/2$ or $T = 365$. These correspond to the first day of summer and the first day of winter. For the northern hemisphere, $\tan(l) > 0$, so we find from the first derivative test that T=0 (or 365) is a local maximum and $T = 365/2$ is a local minimum. Plot the sunset function for Charlotte:

$$S_{\text{Charlotte}} = substitute\left[l = \frac{35.25\pi}{180}, a = \frac{23.5\pi}{180}\right] \text{ into } sunset$$

Then plot this function for S between 0 and 365. Is the graph shaped like you would expect it to be?

The derivative measures the rate of change of sunset with respect to the time of year. How does this change? What are the changes that we might expect? How does the day lengthen or shorten as the time of year changes?

$$\frac{dS}{dT} = -\frac{24}{365}\tan(l)\left[\frac{\sin(\alpha)\sin(2\pi T / 365)}{\left(1-\sin^2(\alpha)\cos^2(2\pi T / 365)\right)\sqrt{1-\sin^2(\alpha)\cos^2(2\pi T / 365)\sec^2(t)}}\right]$$

So when we give the latitude and the number of days after June 21, this will give us the approximate value for the number of minutes later, or earlier, the sun will set the following evening.

If we differentiate the above equation with respect to T, the extreme values for $\frac{dS}{dT}$ occur when $2\pi T/365 = \pi/2$ or $3\pi/2$. When $2\pi T/365 = \pi/2$ (the first day of autumn)

$$\frac{dS}{dT} = -\frac{24}{365}\tan(l)\sin(\alpha),$$

which for Charlotte gives 2.56 minutes per day earlier for the sunset. On the first day of spring, when $2\pi T/365 = 3\pi/2$, the days are lengthening most rapidly with the same rate.

Note how this maximal rate changes with latitude. Near the equator, $\tan(l)$ is small. In fact, on the equator $l = 0°$ and $\tan(l) = 0$. In San Juan, PR, the latitude is $18°27'$ and the maximal rate of change there is .52 minutes per day. This corresponds to the fact that seasons don't make much difference near the equator. As we go north the times increase:

City	Latitude	Maximal Rate min/day
Atlanta, GA	$33°45'10''$	1.05
Washington, DC	$38°53'51''$	1.27
Philadelphia, PA	$39°56'58''$	1.32
New York, NY	$40°45'06''$	1.36
Boston, MA	$42°21'24''$	1.43
Bangor, ME	$44°48'13''$	1.56
Vancouver, BC	$49°18'56''$	1.83
Nome, AK	$64°30'00''$	3.30

Near the poles, $\tan(l)$ is *very large*, so the rate of change is tremendous. This large rate corresponds to the sudden switch from nearly 6 months of sunlight to nearly 6 months of darkness. At the poles, the rate of change is infinite. Of course, in reality the change isn't quite sudden because of the sun's diameter, the fact that the earth isn't a perfect sphere, refraction by the atmosphere, *et. al.*.

Solar Energy

The time of sunrise is the negative of the time of sunset, so we have

$$E = \int_{-S}^{S} \sin(A)\,dt,$$

where $\sin(A)$ comes from above and S is the time we found above, as well.

Exercise 3: *Find this integral.*

We can simplify this expression by using the formula

$$\cos(2\pi S / 24) = -\tan(l)\left(\frac{k}{\sqrt{1-k^2}} / \right).$$

This gives us through some trigonometric identities to:

$$\sin\left(\frac{2\pi S}{24}\right) = \sqrt{\frac{1-(k\cdot\sec(l))^2}{1-k^2}}.$$

Since both k and $\sqrt{1-k^2}$ appear, we can simplify further by writing $k = \sin(D)$ [3].

It would take some time to simplify this. We can see some simple substitutions to make and get:

$$\text{Energy} = \left(\frac{24}{\pi}\right)\cos(l)\sqrt{1-(\sec(l)\sin(D))^2} + \sin(l)\sin(D)\arccos\left(-\tan(l)\tan(D)\right)$$

We have had to ignore the constant in the energy that could only be found empirically, so we can simplify the energy expression a bit more by ignoring the factor of $\frac{24}{\pi}$. Incorporating the $\cos(l)$ into the square root, we get (finally)

$$E = \sqrt{\cos^2(l) - \sin^2(D)} + \sin(l)\sin(D)\arccos\left(-\tan(l)\tan(D)\right),$$

where $\sin(D) = \sin(\alpha)\cos(2\pi T / 365)$.

Thus, the energy is a function of the latitude, l, and the time of year T. You can plot the energy over the whole globe for a whole year using a 3-dimensional plot, possibly in WinPlot:

$$f(x,y) = \left(\frac{24}{\pi}\right)\cos(x)\sqrt{1-\sec^2(x)\sin^2(\alpha)\cos^2\left(\frac{2\pi y}{365}\right)}$$
$$+\sin(x)\sin(\alpha)\cos\left(\frac{2\pi y}{365}\right)\arccos\left[-\tan(x)\tan\left(\arcsin\left(\sin(\alpha)\cos\left(\frac{2\pi y}{365}\right)\right)\right)\right]$$

for $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ and $-182 \le y \le 183$.

Exercise 4: *When does the equator receive the most solar energy? the least?*

At the equator, $l = 0$, what do we have?

$$E = \sqrt{1-\sin^2(D)} = \sqrt{1-\sin^2(\alpha)\cos^2\left(\frac{2\pi T}{365}\right)}.$$

Thus, the energy is largest when $\cos^2(2\pi T / 365) = 0$. That is when $\frac{T}{365} = \frac{1}{4}$ or $\frac{3}{4}$;

[3] The number D is important in astronomy. It is called the *declination*.

that is on the first days of spring and fall. On these days $E = 1$. The energy will be the least on the first days of summer and winter when $\cos^2(2\pi / 365) = 1$ and we have $E = \sqrt{1-\sin^2(\alpha)} = \cos(\alpha) = \cos(23.5^\circ) = 0.917$, or about 92% of the maximum value.

Using this example, we can standardize units in which the solar energy can be measured. One unit of *E* is the total energy received on a square meter at the equator on the first day of spring. All other energies may be expressed in terms of this unit.

Exercise 5: *Find the solar energy received at Charlotte on the first days of fall, spring, summer and winter.*

Exercise 6: *Compare the solar energy received on June 21 at the Arctic Circle ($l = 90° - \alpha$) with that received at the equator and the amount received in Charlotte.*

Exercise 7: *Express the total solar energy received over a whole year at latitude l by using summation notation. Write down an integral which is approximately equal to this sum. Ask your calculator to evaluate it.*

Exercise 8: *Find the total solar energy received at a latitude in the polar region on a day on which the sun never sets.*

Exercise 9: *How do you think the climate of the earth would be affected if the inclination α were to become: (a) $10°$? (b) $40°$? In each case discuss whether the North Pole receives more or less energy during the year than the equator.*

Exercise 10: *Determine whether a square meter at the equator or at the North Pole receives more solar energy (a) during the month of February, (b) during the month of April, (c) during the entire year.*