Chapter 9

The Problem of Finite Integration

When is the antiderivative of an elementary function an elementary function? When can the antiderivative of an elementary function be written in finite terms? Why cant I write down $\int e^{x^2} dx$?

The question arises from elementary differential equations: Solve $\dfrac{dy}{dx} = f(x)$. This really is a number of questions:

1. Does the equation have a solution?
2. Is the solution, if it exists, unique?
3. What relations hold between different solutions?
4. What is the functional form of a solution?

We can answer these for the most part:

1. A solution is
$$y = \int_a^x f(u)\, du.$$
if the definite integral exists.
2. $y + C$ is also a solution, where C is an arbitrary constant.
3. All solutions are of the form $y + C$.

Our interest really lies in #4.

9.1 Elementary Functions

An elementary function is a function of one of the following types:

1. rational functions;

2. algebraic functions, explicit or implicit;

3. the exponential function e^x;

4. the logarithmic function $\log(x)$;

5. finite combinations of the previous classes.

Theorem 6 (Laplaces Theorem (1812)) *The integral of a rational function is always an elementary function. In fact, it is either rational or the sum of a rational function and a finite number of constant multiples of logarithms of rational functions.*

How do we handle the following?

$$\int \frac{(x^2+1)^2+x}{x(x^2+1)}\,dx$$

By Partial Fractions, of course. This is *basic algebra*. It is also wise to use ***Maple*** to do this basic algebra.

$$\begin{aligned}
\int \frac{(x^2+1)^2+x}{x(x^2+1)}\,dx &= \int x\,dx + \int \frac{1}{x}\,dx + \int \frac{1}{x^2+1}\,dx \\
&= \frac{1}{2}x^2 + \log|x| + \int \left(\frac{i}{2(x+i)} - \frac{i}{2(x-i)}\right) dx \\
&= \frac{1}{2}x^2 + \log|x| + \frac{i}{2}\log\left|\frac{x+i}{x-i}\right| + C
\end{aligned}$$

This is probably not what you expected, but it is in the format promised by Laplaces Theorem rationals and logarithms. This is not what we would expect, nor get, from ***Maple***, or is it?

9.1.1 Standard Trigonometric Functions

How do we handle the standard trig functions, such as sine and cosine? They do not appear to be in one of the above forms. Each trigonometric function is definable in terms of complex exponentials and so fall under this theorem:

$$\begin{aligned}
\cos(z) &= \frac{1}{2}\left(e^{iz} + e^{-iz}\right) \\
\sin(z) &= \frac{1}{2}\left(e^{iz} - e^{-iz}\right)
\end{aligned}$$

9.2 Hermite's Algorithm

Lets look at a different example:

$$\int \frac{4x^9 + 21x^6 + 2x^3 - 3x^2 - 3}{(x^7 - x + 1)^2}\, dx$$

This is not something we would like to attack using partial fractions. How do we attack problems of this type?

Hermite showed that the rational part of the integral can be determined without a knowledge of the roots of the denominator, and using only elementary algebra!

If P_1 and P_2 are two polynomials in x which have no common factor, and P_3 any third polynomial, then we can determine two polynomials A_1 and A_2, such that

$$A_1P_1 + A_2P_2 = P_3.$$

This is because the gcd of P_1 and P_2 is 1, so there is always a way to write the gcd as a linear combination of the two polynomials. Since that is true, we can then write any polynomial as an appropriate combination of P_1 and P_2.

To integrate

$$\int \frac{P(x)}{Q(x)}\, dx$$

suppose that $Q(x) = Q_1(x)Q_2(x)^2Q_3(x)^3 \ldots Q_n(x)^n$ where each polynomial has only simple roots and no two of which have a common factor. You can then find B and A_1 so that

$$BQ_1 + A_1Q_2^2Q_3^3 \ldots Q_n^n = P,$$

and therefore so that

$$R(x) = \frac{P}{Q} = \frac{A_1}{Q_1} + \frac{B}{Q_2^2Q_3^3 \ldots Q_n^n}.$$

By repeating this process we arrive at

$$R(x) = \frac{A_1}{Q_1} + \frac{A_2}{Q_2^2} + \frac{A_3}{Q_3^3} + \cdots + \frac{A_n}{Q_n^n}$$

and the problem of integrating the rational function is reduced to integrating $\frac{A}{Q^n}$ where Q is a polynomial all of whose roots are distinct. Such a polynomial Q is called *squarefree*.

Since Q is squarefree, Q and Q' have no factors in common. Therefore, there are polynomials C and D so that

$$CQ + DQ' = A$$

Hence,

$$\begin{aligned}
\int AQ^n \, dx &= \int \frac{CQ + DQ'}{Q^n} \, dx \\
&= \int \frac{C}{Q^{n-1}} \, dx - \frac{1}{n-1} \int D \frac{d}{dx} \left(\frac{1}{Q^{n-1}} \right) dx \\
&= -\frac{D}{(n-1)Q^{n-1}} + \frac{1}{n-1} \int \frac{D'}{Q^{n-1}} \, dx + \int \frac{C}{Q^{n-1}} \, dx \\
&= -\frac{D}{(n-1)Q^{n-1}} + \int \frac{E}{Q^{n-1}} \, dx
\end{aligned}$$

where

$$E = C + \frac{D'}{n-1}.$$

So we have reduced the degree of the denominator by one, and have arrived at a similar integral. We can proceed in this way and reduce the degree of $1/Q$ by one at each step. Finally we will arrive at an equation:

$$\int \frac{A}{Q^n} \, dx = R_n(x) + \int \frac{S}{Q} \, dx$$

where R_n is a rational function and S is a polynomial.

The integral on the right hand side has no rational part, since all of the roots of Q are simple (or Q is squarefree). Thus the rational part of $\int R(x) \, dx$ is

$$R_2(x) + R_3(x) + \cdots + R_t(x).$$

Example 9.2.1 So to integrate

$$\int \frac{4x^9 + 21x^6 + 2x^3 - 3x^2 - 3}{(x^7 - x + 1)^2} \, dx$$

take $P_1 = x^7 - x + 1$, $P_2 = P_1' = 7x^6 - 1$, $P_3 = 4x^9 + 21x^6 + 2x^3 - 3x^2$ We have to have $C(x^7 - x + 1) + D(7x^6 - 1) = 4x^9 + 21x^6 + 2x^3 - 3x^2 - 3$ So we can take C to have degree no more than 5 and D to have degree no more than 6. We then get a system of 13 equations in 13 unknowns, which we can solve. In this case we find that:

$$\begin{aligned}
&C = a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 \\
&D = b_6x^6 + b_5x^5 + b_4x^4 + b_3x^3 + b_2x^2 + b_1x + b_0 \\
&C \times (x^7 - x + 1) + D \times (7x^6 - 1) = 4x^9 + 21x^6 + 2x^3 - 3x^2 - 3 \\
&(a_5 + 7b_6)\, x^{12} + (a_4 + 7b_5)\, x^{11} + (a_3 + 7b_4)\, x^{10} + (a_2 + 7b_3)\, x^9 + (a_1 + 7b_2)\, x^8 + (a_0 + 7b_1)\, x^7 + \\
&\quad (7b_0 - b_6 - a_5)\, x^6 + (a_5 - b_5 - a_4)\, x^5 + (a_4 - b_4 - a_3)\, x^4 + (a_3 - b_3 - a_2)\, x^3 + (a_2 - b_2 - a_1)\, x^2 + \\
&\quad (a_1 - b_1 - a_0)\, x + (a_0 - b_0) = 4x^9 + 21x^6 + 2x^3 - 3x^2 - 3
\end{aligned}$$

Giving us the system of equations

$$\begin{aligned}
a_5 + 7b_6 &= 0\\
a_4 + 7b_5 &= 0\\
a_3 + 7b_4 &= 0\\
a_2 + 7b_3 &= 4\\
a_1 + 7b_2 &= 0\\
a_0 + 7b_1 &= 0\\
7b_0 - b_6 - a_5 &= 21\\
a_5 - b_5 - a_4 &= 0\\
a_4 - b_4 - a_3 &= 0\\
a_3 - b_3 - a_2 &= 2\\
a_2 - b_2 - a_1 &= -3\\
a_1 - b_1 - a_0 &= 0\\
a_0 - b_0 &= -3
\end{aligned}$$

We have to solve this system of 13 equations in 13 unknowns. This gives:

$$\begin{aligned}
&a_5 = a_4 = a_2 = a_1 = a_0 = 0\\
&a_3 = -3\\
&b_6 = b_5 = b_4 = b_2 = b_1 = 0\\
&b_3 = 1, \quad b_0 = 3
\end{aligned}$$

Thus,

$$C = -3x^2, \quad D = x^3 + 3$$

According to the Hermite Algorithm then we have

$$\int \frac{4x^9 + 21x^6 + 2x^3 - 3x^2 - 3}{(x^7 - x + 1)^2}\, dx = -\frac{x^3 + 3}{x^7 - x + 1}$$

$$S = -3x^2 + \tfrac{d}{dx}\left(x^3 + 3\right) = 0$$

Thus, this is the complete integral. Note that this is mostly algebra! Comparing this on ***Maple***, ***Mathematica***, ***Derive***, and the ***TI-89***, we get:

Maple	***Mathematica***	***Derive***	***TI-89***
$\dfrac{-3 - x^3}{x^7 - x + 1}$	$-\dfrac{x^3 + 3}{x^7 - x + 1}$	no response	no response

Example 9.2.2

$$\int R(x)\,dx = \int \frac{x^7 - 24x^4 - 4x^2 + 8x - 8}{x^8 + 6x^6 + 12x^4 + 8x^2}\,dx$$

We need to factor the denominator:

$$Q = x^8 + 6x^6 + 12x^4 + 8x^2 = x^2(x^2+2)^3 = Q_2^2 Q_3^3$$

R now decomposes to:

$$R = \frac{x-1}{x^2} + \frac{x^4 - 6x^3 - 18x^2 - 12x + 8}{(x^2+2)^3} = \frac{A_2}{Q_2^2} + \frac{A_3}{Q_3^3}$$

$$\int R(x)\,dx = \int \frac{A_2}{Q_2^2}\,dx + \int \frac{A_3}{Q_3^3}\,dx = \int \frac{x-1}{x^2}dx + \int \frac{x^4 - 6cx^3 - 18x^2 - 12x + 8}{(x^2+2)^3}\,dx$$

Applying Hermite's Algorithm to the first integral we must find C and D so that:

$$Cx + D = x - 1$$

$$C = 1, \qquad D = -1$$

$$\int \frac{x-1}{x^2}dx = \frac{1}{x} + \int \frac{dx}{x}$$

For the second integral we need to find C and D so that

$$C\left(x^2+2\right) + D\,(2x) = x^4 - 6x^3 - 18x^2 - 12x + 8$$

Setting $C = a_2x^2 + a_1x + a_0$ and $D = b_3x^3 + b_2x^2 + b_1x + b_0$ we get

$$(a_2x^2 + a_1x + a_0)(x^2+2) + (b_3x^3 + b_2x^2 + b_1x + b_0)(2x) = x^4 - 6x^3 - 18x^2 - 12x + 8$$

which gives

$$\begin{aligned} a_2 + 2b_3 &= 1 \\ a_1 + 2b_2 &= -6 \\ 2a_2 + a_0 + 2b_1 &= -18 \\ 2a_1 + 2b_0 &= -12 \\ 2a_0 &= 8 \end{aligned}$$

One solution to this over-determined system is

$$a_0 = 4, \quad a_1 = -6, \quad a_2 = 1, \quad b_3 = b_2 = b_0 = 0, \quad b_1 = -12.$$

$$\begin{aligned} \int \frac{x^4 - 6x^3 - 18x^2 - 12x + 8}{(x^2+2)^3}dx &= -\frac{1}{2}\frac{D}{(x^2+2)^2} + \int \frac{C + 1/2D'}{(x^2+2}dx \\ &= \frac{6x}{(x^2+2)^2} + \int \frac{x^2 - 6x - 2}{(x^2+2)}dx \end{aligned}$$

Repeating the process on the above integral gives us the following result:

$$\int \frac{x^7 - 24x^4 - 4x^2 + 8x - 8}{x^8 + 6x^6 + 12x^4 + 8x^2}\, dx = \frac{1}{x} + \frac{6x}{(x^2+2)^2} - \frac{x-3}{(x^2+2)} + \int \frac{dx}{x}$$

The partial fractional decomposition of $R(x)$ is:

$$-\frac{1}{x^2} + \frac{1}{x} + \frac{1}{x^2+2} - 2\frac{3x+11}{(x^2+2)^2} + \frac{48}{(x^2+2)^3}$$

9.3 Horowitz-Ostrogradsky Algorithm

Ostrogradskys algorithm was known in the mid 1800's, but was not widely used. It also computes the rational part of the integral, but it reduces to solving systems of linear algebraic equations instead of solving polynomial Diophantine equations as in the previous example. Start as before with

$$R = \frac{A}{Q}, \quad Q = Q_1 Q_2^2 Q_3^3 \ldots Q_n^n$$

and let

$$Q^{\#} = \gcd(Q, Q'), \quad Q^* = Q/Q^{\#}.$$

We can find polynomials B and C so that

$$A = B'Q^* - B\left(\frac{Q^* Q^{\#\prime}}{Q^{\#}}\right) + CQ^{\#},\ \deg(B) = \deg(Q^{\#}) - 1, \deg(C) = \deg(Q^{*)-1}$$

which reduces to a system of linear equations! The solution to the integration problem is then:

$$\int \frac{A}{Q} = \int \frac{B'Q^* - B\left(\frac{Q^* Q^{\#\prime}}{Q^{\#}}\right)}{Q} + \int \frac{CQ^{\#}}{Q} = \int \left(\frac{B'Q^{\#} - BQ^{\#\prime}}{(Q^{\#})^2}\right) + \int \frac{C}{Q^*}$$

$$\int \frac{A}{Q} = \frac{B}{Q^{\#}} + \int \frac{C}{Q^*}$$

So

$$Q^{\#} = \gcd(Q, Q') = x^5 + 4x^3 + 4$$

$$Q^* = Q/Q^{\#} = x^3 + 2x$$

$$n = \deg(Q^{\#}) - 1 = 4, \quad m = \deg(Q^*) - 1 = 2$$

$$\begin{aligned} H = A - Q^* \frac{d}{dx}\left(\sum_{i=0}^{n} b_i x^i\right) + \left(\sum_{i=0}^{n} b_i x^i\right)\frac{Q^* Q^{\#\prime}}{Q^{\#}} - Q^{\#}\left(\sum_{j=0}^{m} c_j x^j\right) \\ = (1 - c_2)\, x^7 + (b_4 - c_1)\, x^6 + (2b_3 - c_0 - 4c_2)\, x^5 \\ + (3b_2 - 6b_4 - 4c_1 - 24)\, x^4 + 4\,(b_1 - b_3 - c_0 - c_2)\, x^3 \\ + (5b_0 - 2b_2 - 4c_1 - 4)\, x^2 + 4\,(2 - c_0)\, x + 2\,(b_0 - 4) \\ = 0 \end{aligned}$$

This system has a unique solution

$$(b_0, b_1, b_2, b_3, b_4, c_0, c_1, c_2) = (4, 6, 8, 3, 0, 2, 0, 1)\,.$$

This gives us as a solution to the integration problem

$$\begin{aligned}\int \frac{x^7 - 24x^4 - 4x^2 + 8x - 8}{x^8 + 6x^6 + 12x^4 + 8x^2}\,dx &= \frac{3x^3 + 8x^2 + 6x + 4}{x^5 + 4x^3 4x} + \int \frac{x^2 + 2}{x^3 + 2x}\,dx \\ &= \frac{3x^3 + 8x^2 + 6x + 4}{x^5 + 4x^3 4x} + \int \frac{dx}{x}\end{aligned}$$

Maple	$\frac{1}{x} - \frac{1}{4}\frac{22x - 12}{x^2 + 2} + 6\frac{x}{(x^2 + 2)^2} + \frac{9}{2}\frac{x}{x^2 + 2} + \int \frac{dx}{x}$
Mathematica	$\frac{1}{x} + \frac{6x}{(2 + x^2)^2} + \frac{3 - x}{2 + x^2} + \int \frac{dx}{x}$
Derive	$\frac{1}{x} - \frac{x^3 - 3x^2 - 4x - 6}{(x^2 + 2)^2} + \int \frac{dx}{x}$
TI-89	$\int \frac{dx}{x} - \frac{x^3 - 3x^2 - 4x - 6}{(x^2 + 2)^2} + \frac{1}{x}$

9.4 Extensions

There was a lot of work done on this problem in the nineteenth century. They were able to show

1. If the integral of an algebraic function is elementary, then it must be of the form

$$\int y\,dx = R_0(x) + \sum_k c_k \log\left(R_k(x)\right)$$

   where the R_k are all rational functions and the number of terms in the sum is finite but undetermined.

2. If the integral $I = \int f\,e^g dx$ is elementary, and f and g are elementary then I is of the form $I = Re^g$, where R is a rational function of x, f, and g.

This last one is sufficient to show why there is no elementary formula for $\int e^{x^2}\,dx$. If there is an elementary functional antiderivative, then $I = \int e^{x^2} dx = R \cdot e^{x^2}$ where R is a rational function of x. Differentiating both sides we get

$$e^{x^2} = I' = R'\,e^{x^2} + 2xR\,e^{x^2}$$

Then, equating coefficients

$$R' + 2xR = 1$$

R is a rational function, say N/D, so by the quotient rule

$$\frac{N'D - ND'}{D^2} + 2x\frac{N}{D} = 1$$
$$N'D - ND' + 2xND = D^2$$

The degrees of these expressions are $d+n-1$, $d+n-1$, $d+n+1$, and d^2, respectively. The first two terms cannot equal one another unless $N = D$, which implies that R is a constant. We cannot balance this equation, so no values of n and d will work. Thus, I does not have an elementary antiderivative.

9.5 Further Extensions

In 1834 Liouville extended Laplaces theorem to algebraic functions if they have an elementary antiderivative.

Theorem 7 (Liouville, 1834) *If $f(x)$ is an algebraic function of x and if $\int f(x)\,dx$ is elementary, then*

$$\int f(x)\,dx = U_0 + \sum_{j=1}^{n} C_j \ln(U_j)$$

where the C_j's are constants and the U_j's are algebraic functions of x.

Theorem 8 (Strong Liouville Theorem, 1835) *(a) If F is an algebraic function of x, $y_1,\ldots, y_m$, where $y_1,\ldots,y_m$ are functions of x whose derivatives $dy_1/dx, \ldots, dy_m/dx$ are each algebraic functions of x, $y_1,\ldots, y_m$, then $\int F(x, y_1, \ldots, y_m)\,dx$ is elementary if and only if*

$$\int F(x, y_1, \ldots, y_m)\,dx = U_0 + \sum_{j=1}^{n} C_j \ln(U_j)$$

where the C_j's are constants and the U_j's are algebraic functions of $x, y_1, \ldots, y_m$.

(b) If $F(x, y_1, \ldots, y_m)$ is a rational function and $dy_1/dx, \ldots, dy_m/dx$ are rational functions of $x, y_1, \ldots, y_m$, then the U_j's in part (a) must be rational functions of $x, y_1, \ldots, y_m$.

From this Liouville proved the following:
If $f(x)$ and $g(x)$ are rational functions with $g(x)$ nonconstant, then $\int f(x)e^{g(x)}dx$ is elementary if and only if there exists a rational function $R(x)$ such that $f(x) = R'(x) + R(x)g'(x)$.

Using this you can show that the following integrals are not elementary:

$$\int x^{2n} e^{ax^2}, \quad \int x^{-n} e^{cx}$$

$$\int \sqrt{\log x} = \int 2t^2 e^{t^2}, \quad \text{where} \quad t^2 = \log x$$
$$\int \frac{1}{\sqrt{\log x}} = \int 2e^{t^2}, \quad \text{where} \quad t^2 = \log x$$
$$\int \frac{e^{ax}}{\sqrt{x}} = \int 2e^{at^2}, \quad \text{where} \quad t^2 = x$$
$$\int e^{e^x} = \int \frac{e^t}{t}, \quad \text{where} \quad t = e^x$$
$$\int \frac{1}{\log x} = \int \frac{e^t}{t}, \quad \text{where} \quad t = \log x$$

Hardy extended Liouville's work and in 1905 proved the following:
Liouville-Hardy (1905): *If $f(x)$ is a rational function, then $\int f(x)\log(x)dx$ is elementary if and only if there exists a rational function $g(x)$ and a constant C such that $f(x) = C/x + g'(x)\log(x)$.*

Independently of the others the Russian mathematician Chebyshev worked on the question of which arc lengths and surface areas are integrable in finite terms. He arrived at the following result:
Chebyshev's Theorem (1853) *If p, q, and r are rational numbers and a and b are real numbers with $abr \neq 0$, then $\int x^p(a + bx^r)^q\,dx$ is elementary if and only if at least one of $(p+1)/r$, q, or $(p+1)/r + q$ is an integer.*

$$\int \sqrt[3]{1 + x^2}dx$$

is not elementary. Here, $p = 0$, $r = 2$, $q = 1/3$, so $(p+1)/r = 1/2$, $q = 1/3$, $(p+1)/r+q = 5/6$.

Example 9.5.1 Consider the arc length of the graph of $f(x) = x^k$, given by $\int \sqrt{1 + k^2x^{2k-2}}\,dx$. This integral is elementary if and only if either $1/(2k-2)$ or $1/(2k-2) + 1/2$ is an integer, where $k \neq 1$. Thus, the related arc length integral for $f(x) = x^k$ is elementary if and only if $k = 1$ or $k = 1 + 1/n$, where n is an integer. If follows that, for example, $\int \sqrt{1 + x^3}\,dx$ and $\int \sqrt{1 + x^{-4}}\,dx$ are nonelementary integrals. [This last integral is the arc length integral for $f(x) = 1/x$.]

Example 9.5.2 A similar calculation can be performed for integrals representing the area of the surface obtained by revolving the graph of $f(x) = x^k$ about the x-axis. These integrals are elementary if and only if $k = 1$ or $k = 1 + 2/n$, where n is an integer.