Appendix C

Irrationality of e

The series expansion of the number e is given by

$$e = \sum_{n=0}^{\infty} \frac{1}{n!},$$

which we will prove later. We shall use this series representation of e to show that it is irrational. The proof is due to Joseph Fourier.

Theorem C.1 *e is irrational.*

Assume not, that is, assume that e is rational. Suppose $e = a/b$, for some positive integers a and b. Construct the number

$$x = b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right).$$

First we will show that x is an integer, then show that x is less than 1 and positive. This contradiction will establish the irrationality of e.

To see that x is an integer, note that

$$\begin{aligned}
x &= b!\left(e - \sum_{n=0}^{b} \frac{1}{n!}\right) \\
&= b!\left(\frac{a}{b} - \sum_{n=0}^{b} \frac{1}{n!}\right) \\
&= a(b-1)! - \sum_{n=0}^{b} \frac{b!}{n!} \\
&= a(b-1)! - \sum_{n=0}^{b} \frac{1 \cdot 2 \cdot 3 \cdots (n-1)(n)(n+1) \cdots (b-1)(b)}{1 \cdot 2 \cdot 3 \cdots (n-1)(n)} \\
&= a(b-1)! - \sum_{n=0}^{b} (n+1)(n+2) \cdots (b-1)(b)
\end{aligned}$$

and this last expression is an expression involving only integers. Thus, since x is a sum of integers, it is an integer.

Clearly, x is positive since $x = b! \sum_{n=b+1}^{\infty} \frac{1}{n!} > 0$. Now, notice that

$$\begin{aligned} x &= b! \sum_{n=b+1}^{\infty} \frac{1}{n!} \\ &= \frac{1}{b+1} + \frac{1}{(b+1)(b+2)} + \frac{1}{(b+1)(b+2)(b+3)} + \cdots \\ &< \frac{1}{b+1} + \frac{1}{(b+1)^2} + \frac{1}{(b+1)^3} + \cdots \\ &= \frac{1}{(b+1)-1} = \frac{1}{b} \\ &\leq 1. \end{aligned}$$

Now, if $x = 1$, then $b = 1$ and $e = \frac{a}{b} = a$ is an integer, which we know is not true. Thus, $0 < x < 1$ and x is an integer. This contradiction gives us then that e is irrational.