Appendix D

Irrationality of π

D.1 First Proof

For any integer n and real number r we can define a quantity A_n by the definite integral

$$A_n = \int_{-1}^{1} (1 - x^2)^n \cos(rx)\, dx.$$

If you integrate this by parts you find that the quantities A_n for $n = 2, 3, 4, \ldots$ satisfy the recurrence relation

$$A_n = \frac{2n(2n-1)A_{n-1} - 4n(n-1)A_{n-2}}{r^2}.$$

We can obviously express A_{n-1} and A_{n-2} in terms of lower members of the recurrence, and so on, all the way back to A_0 and A_1. The result is

$$A_n = \frac{n!}{r^{2n+1}} \left[P(r)\sin(r) - Q(r)\cos(r) \right],$$

where $P(r)$ and $Q(r)$ are polynomials in r of degree less than $2n-1$ with integer coefficients.

Now assume that $\pi = a/b$, where a and b are integers, and let's set $r = \pi/2$. Of course, this means that $r = a/(2b)$. If we substitute this value into the preceding equation, and remember that $\sin(\pi/2) = 1$ and $\cos(\pi/2) = 0$, we have

$$\left(\frac{a}{2b}\right)^{2n+1} A_n = n! P(a/2b).$$

Multiplying both sides by $(2b)^{2n+1}$ and dividing by $n!$ gives

$$\frac{a^{2n+1} A_n}{n!} = (2b)^{2n+1} P(a/2b).$$

Remember that $P(a/2b)$ is a polynomial in $a/2b$ with integer coefficients, and its degree is less than $2n+1$. Therefore, when we multiply through by the factor $(2b)^{2n+1}$ we clear out all the $2b$'s in the denominators, so the quantity on the right hand side of the preceding equation is clearly an integer. Thus the left side must also be an integer.

Now, recall that A_n was defined as the integral

$$A_n = \int_{-1}^{1} (1 - x^2)^n \cos(rx)\, dx.$$

Clearly the leading factor of this integrand, $(1 - x^2)^n$, is always between 0 and 1, so an upper bound on the value of A_n is given by

$$A_n \leq \int_{-1}^{1} \cos(rx)\, dx.$$

This integral has some constant value C, though it doesn't matter what it is, and this is the upper bound on the value of A_n for any n. Now we showed previously that the quantity

$$\frac{a^{2n+1} A_n}{n!}$$

is always an integer (assuming π is rational). Since A_n is less than C, it follows that this integer given by the above expression is less than

$$\frac{a^{2n+1} C}{n!}.$$

This is impossible, because $n!$ increases faster than a^{2n+1}, so there is some value of n beyond which this ratio will be less than 1. This proves the quantity can't be an integer, so we have a contradiction. We are forced to conclude that π is not a rational number.

By the way, the actual expressions for the first few A_n and the polynomials $P(r)$ and $Q(r)$ are

$$\begin{aligned}
A_0 &= \frac{2}{r} \sin(r) \\
A_1 &= \frac{4}{r^3} \left(1 \cdot \sin(r) - r \cdot \cos(r)\right) \\
A_2 &= \frac{16}{r^5} \big((3 - r^2) \sin(r) - (3r) \cos(r)\big) \\
A_3 &= \frac{96}{r^7} \big((15 - 6r^2) \sin(r) - (15r - r^3) \cos(r)\big) \\
A_4 &= \frac{768}{r^9} \big((105 - 45r^2 + r^4) \sin(r) - (105r - 10r^3) \cos(r)\big)
\end{aligned}$$

and so on.

The double factorial of a positive integer n is a generalization of the usual factorial $n!$ defined by

$$n!! = \begin{cases} n\cdot(n-2)\dots 5\cdot 3\cdot 1 & \text{if } n>0 \text{ is odd} \\ n\cdot(n-2)\dots 6\cdot 4\cdot 2 & \text{if } n>0 \text{ is even} \\ 1 & n=-1,0 \end{cases}$$

For $n = 0, 1, 2, \dots$ the first few values are $1, 1, 2, 3, 8, 15, 48, 105, 384, \dots$.

The general expression for A_n is

$$A_n = \frac{2^{n+1}n!}{a^{2n+1}}(P_n(r)\sin(r) - Q_n(r)\cos(r))$$

where the coefficient of r^{2k} in the polynomial P_n is

$$(-1)^k\binom{n-k}{k}\frac{(2n-2k-1)!!}{(2k-1)!!}.$$

For example, to determine the coefficient of r^2 in A_4 we have $k = 1$ and $n = 4$, which gives

$$(-1)^1\binom{3}{1}\frac{5!!}{1!!} = (-1)(3)(15) = -45$$

Of course, the coefficients of odd powers of r in P_n are all zero, as are the coefficients of all even powers of r in Q_n. The coefficient of r^{2k+1} in Q_n is

$$(-1)^k\binom{n-k-1}{k}\frac{(2n-2k-1)!!}{(2k+1)!!}.$$

For example, the coefficient of r^3 in A_4 is found by setting $k = 1$ and $n = 4$ to give

$$(-1)^1\binom{2}{1}\frac{5!!}{3!!} = (-1)(2)(15/3) = -10.$$

D.2 Second Proof

Define a function $f_n(x)$ by

$$f_n(x) = \frac{x^n(1-x)^n}{n!}.$$

Lemma D.1 *This function has the following properties:*

1. $0 < f_n(x) < 1/n!$ *for* $0 < x < 1$*;*
2. $f_n^{(k)}(0)$ *and* $f_n^{(k)}(1)$ *are both integers;*

where $f_n^{(k)}(x)$ *denotes the* k^{th} *derivative of* f_n *with respect to* x*.*

To see this, use the binomial theorem and notice that when the numerator is multiplied out, the lowest power of x will be n and the highest power of x will be $2n$. Thus, the function can be written as

$$f_n(x) = \frac{1}{n!}\sum_{k=n}^{2n} c_k x^k$$

where all of the coefficients are integers. In this representation it is clear that $f_n^{(k)}(0) = 0$ for $k < n$ or $k > 2n$.

Considering the sum again, we see that

$$\begin{aligned} f_n^{(n)}(x) &= \frac{1}{n!}[n!c_n + \text{terms involving } x] \\ f_n^{(n+1)}(x) &= \frac{1}{n!}[(n+1)!c_{n+1} + \text{terms involving } x] \\ &\vdots \\ f_n^{(2n)}(x) &= \frac{1}{n!}[(2n)!c_{2n}] \end{aligned}$$

so this implies that $f_n^{(k)}(0)$ is an integer for any k.

Moreover, since

$$f_n^{(k)}(x) = (-1)^k f_n(k)(1-x)$$

and we have

$$f_n(x) = f_n(1-x)$$

so $f_n^{(k)}(1)$ is an integer for any k.

One further fact to note is that $\left|\frac{a^n}{n!}\right| < \epsilon$ for n sufficiently large, say $n > 2a$, and any a.

Theorem D.1 *π is irrational.*

This approach is due to Legendre. We will prove that π^2 is not rational, which implies that π is not rational.

Assume that π^2 is rational and let $\pi^2 = a/b$ for two integers a and b.

Define the function

$$G(x) = b^n\left[\pi^{2n} f_n(x) - \pi^{2n-2} f_n^{(2)}(x) + \cdots + (-1)^n f_n^{(2n)}(x)\right].$$

Each of the factors

$$b^n \pi^{2n-2k} = b^n \left(\frac{a}{b}\right)^{n-k} = a^{n-k} b^k$$

is an integer. We also know that $f_n^{(k)}(0)$ and $f_n^{(k)}(1)$ are integers as well. Therefore, $G(0)$ and $G(1)$ are integers.

Differentiating G twice we get

$$G''(x) = b^n \left[\pi^{2n} f_n^{(2)}(x) - \pi^{2n-2} f_n^{(4)}(x) + \cdots + (-1)^n f_n^{(2n)}(x)\right],$$

since the last term in the equation is the $(2n+2)$-th derivative of f_n is 0. Now, adding $G(x)$ and $G''(x)$ we get

$$G(x) + \pi^2 G''(x) = b^n \pi^{2n+2} f_n(x) = \pi^2 a^n f_n(x).$$

Define a second function by

$$H(x) = G'(x)\sin(\pi x) - \pi G(x)\cos(\pi x).$$

Using the above equation and differentiating H, we have

$$\begin{aligned} H'(x) &= \pi G'(x)\cos(\pi x) + G''(x)\sin(\pi x) - \pi G'(x)\cos(\pi x) + \pi^2 G(x)\sin(\pi x) \\ &= [G''(x) + \pi^2 G(x)]\sin(\pi x) \\ &= \pi^2 a^n f_n(x)\sin(\pi x) \end{aligned}$$

By the Second Fundamental Theorem of Calculus

$$\begin{aligned} \pi^2 \int_0^1 a^n f_n(x)\sin(\pi x)\,dx &= H(1) - H(0) \\ &= G'(1)\sin(\pi) - \pi G(1)\cos(\pi) - G'(0)\sin(0) + \pi G(0)\cos(0) \\ &= \pi[G(1) + G(0)] \end{aligned}$$

Thus, the integral

$$\pi \int_0^1 a^n f_n(x)\sin(\pi x)\,dx$$

is an integer. But we also know that $0 < f_n(x) < 1/n!$ for $0 < x < 1$. Therefore, estimating the above integral, we get

$$0 < \pi a^n f_n(x)\sin(\pi x) < \frac{a^n}{n!}$$

for $0 < x < 1$.

Therefore, we can estimate our integral to get

$$0 < \pi \int_0^1 a^n f_n(x)\sin(\pi x)\,dx < \frac{a^n}{n!} < 1.$$

Here we have used the fact that the last fraction approaches zero if n is sufficiently large. But now we have a contradiction, because that integral is an integer. Since there is no positive integer less than 1, our assumption that π^2 was rational resulted in a contradiction. Hence, π^2 must be irrational.