

# Appendix E

## Quartic Formula

$$\begin{aligned}
r_1 = & -\frac{a}{4} - \frac{1}{2} - \frac{2b}{3} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \\
& + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \\
& - \frac{1}{2} \left[ \frac{a^2}{2} - \frac{4b}{3} - \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \right. \\
& \left. \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{3}} \right. \\
& \left. - a^3 + 4ab - 8c \right. \\
& \left. \sqrt[4]{\frac{a^2}{4} - \frac{2b}{3} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}}} \right.} \\
& \left. \right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
r_2 = & -\frac{a}{4} - \frac{1}{2} \left[ \frac{a^2}{4} - \frac{2b}{3} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \right]^{\frac{1}{3}} \\
& + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \\
& + \frac{1}{2} \left[ \frac{a^2}{2} - \frac{4b}{3} - \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \right]^{\frac{1}{3}} \\
& - \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \\
& \left. \sqrt[4]{\frac{a^2 - 2b}{4} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}}} \right]^{\frac{1}{2}}
\end{aligned}$$

$$\begin{aligned}
r_3 = & -\frac{a}{4} + \frac{1}{2} \left[ \frac{a^2}{4} - \frac{2b}{3} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \right]^{\frac{1}{3}} \\
& + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \\
& - \frac{1}{2} \left[ \frac{a^2}{2} - \frac{4b}{3} - \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \right]^{\frac{1}{3}} \\
& - \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \\
& \left. \left[ \frac{a^2 - 2b}{4} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} \right. \\
& \left. - \left[ \frac{a^2 - 2b}{4} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \right]^{\frac{1}{3}} \right]
\end{aligned}$$

$$\begin{aligned}
r_4 = & -\frac{a}{4} + \frac{1}{2} \left[ \frac{a^2}{4} - \frac{2b}{3} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \right]^{\frac{1}{3}} \\
& + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \\
& + \frac{1}{2} \left[ \frac{a^2}{2} - \frac{4b}{3} - \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} \right]^{\frac{1}{3}} \\
& - \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}} \\
& \left. \sqrt[4]{\frac{a^2 - 2b}{4} + \frac{\sqrt[3]{2}(b^2 - 3ac + 12d)}{3 \left( 2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2} \right)^{\frac{1}{3}}} + \left( \frac{2b^3 - 9abc + 27c^2 + 27a^2d - 72bd + \sqrt{-4(b^2 - 3ac + 12d)^3 + (2b^3 - 9abc + 27c^2 + 27a^2d - 72bd)^2}}{54} \right)^{\frac{1}{2}}} \right]^{\frac{1}{2}} \\
& - a^3 + 4ab - 8c
\end{aligned}$$