Appendix F

The Gamma Function

In mathematics, the Gamma function extends the factorial function to complex and non-integer numbers. The factorial function is defined for positive integers. The Gamma function extends the factorial function to non-integer and complex values of n. If z is a real variable, then only when z is a natural number, we have

$$\Gamma(z+1) = z!$$

but for non-natural values of z, the above equation does not apply, since the factorial function is not defined.

F.1 Definition

The notation $\Gamma(z)$ is due to Adrien-Marie Legendre. If the real part of the complex number z is positive, then the integral

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt$$

converges absolutely. Using integration by parts, one can show that

$$\Gamma(z+1) = z\Gamma(z). \tag{F.1}$$

This functional equation reveals the connection with the factorial function. Because $\Gamma(1) = 1$, this relation implies that

$$\Gamma(n+1) = n\Gamma(n) = n(n-1)\Gamma(n-1) = \cdots = n!\Gamma(1) = n!.$$

for all natural numbers n.

Euler did not define it by an integral, though. The following infinite product definition for the Gamma function is due to Euler. It is valid for all complex numbers z which are not negative integers or zero

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!n^z}{z(z+1)\cdots(z+n)}.$$

Weierstrauss extended this definition to

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right)^{-1} e^{z/n},$$

where γ is the Euler-Mascheroni constant.

It is straightforward to show that the Euler definition satisfies the functional equation above. Provided z is not equal to 0, -1, -2 ...

$$\begin{aligned}
\Gamma(z+1) &= \lim_{n\to\infty} \frac{n!n^{z+1}}{(z+1)(z+2)\cdots(z+1++n)} = \lim_{n\to\infty} \left(z \frac{n!n^z}{z(z+1)\cdots(z+n)} \frac{n}{(z+1+n)} \right) \\
&= z\Gamma(z) \lim_{n\to\infty} \frac{n}{(z+1+n)} \\
&= z\Gamma(z)
\end{aligned}$$

F.2 Properties

Other important functional equations for the Gamma function are Euler's reflection formula

$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}$$

and the duplication formula

$$\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$

The duplication formula is a special case of the multiplication theorem

$$\Gamma(z)\Gamma\left(z + \frac{1}{m}\right)\Gamma\left(z + \frac{2}{m}\right)\ldots\Gamma\left(z + \frac{m-1}{m}\right) = (2\pi)^{(m-1)/2} m^{1/2-mz}\Gamma(mz).$$

Perhaps the most well-known value of the Gamma function at a non-integer argument is

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi},$$

which can be found by setting z=1/2 in the reflection formula. In general, for odd integer values of n we have:

$$\Gamma\left(\frac{n}{2} + 1\right) = \sqrt{\pi}\frac{n!!}{2^{(n+1)/2}}, \ (n \text{ odd})$$

where $n!!$ denotes the double factorial

$$n!! = \begin{cases} 1, & \text{if } n = 0 \text{ or } n = 1; \\ n(n-2)! & \text{if } n \geq 2. \end{cases}$$

Note that this definition can be used to define double factorials of negative integers:

$$(n-2)!! = \frac{n!!}{n}$$

The sequence of double factorials for $n = -1, -3, -5, -7, \ldots$ starts as

$$1, -1, 1/3, -1/15, \ldots$$

while the double factorial of negative even integers is infinity.

Some identities involving double factorials are:

$$\begin{aligned}
n! &= n!!(n-1)!! \\
(2n)!! &= 2^n n! \\
(2n+1)!! &= \frac{(2n+1)!}{(2n)!!} = \frac{(2n+1)!}{2^n n!} \\
\Gamma\left(n+\frac{1}{2}\right) &= \sqrt{\pi}\frac{(2n-1)!!}{2^n} \\
\Gamma\left(\frac{n}{2}+1\right) &= \sqrt{\pi}\frac{n!!}{2^{(n+1)/2}}
\end{aligned}$$

F.3 Particular values of the Gamma function

$$\begin{aligned}
\Gamma\left(-\frac{3}{2}\right) &= \frac{4\sqrt{\pi}}{3} \\
\Gamma\left(-\frac{1}{2}\right) &= -2\sqrt{\pi} \\
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi} \\
\Gamma(1) &= 0! = 1 \\
\Gamma\left(\frac{3}{2}\right) &= \frac{\sqrt{\pi}}{2} \\
\Gamma(2) &= 1! = 1 \\
\Gamma\left(\frac{5}{2}\right) &= \frac{3\sqrt{\pi}}{4} \\
\Gamma(3) &= 2! = 2 \\
\Gamma\left(\frac{7}{2}\right) &= \frac{15\sqrt{\pi}}{8} \\
\Gamma(4) &= 3! = 6
\end{aligned}$$

For real values of z, a plot of the Gamma function looks like:

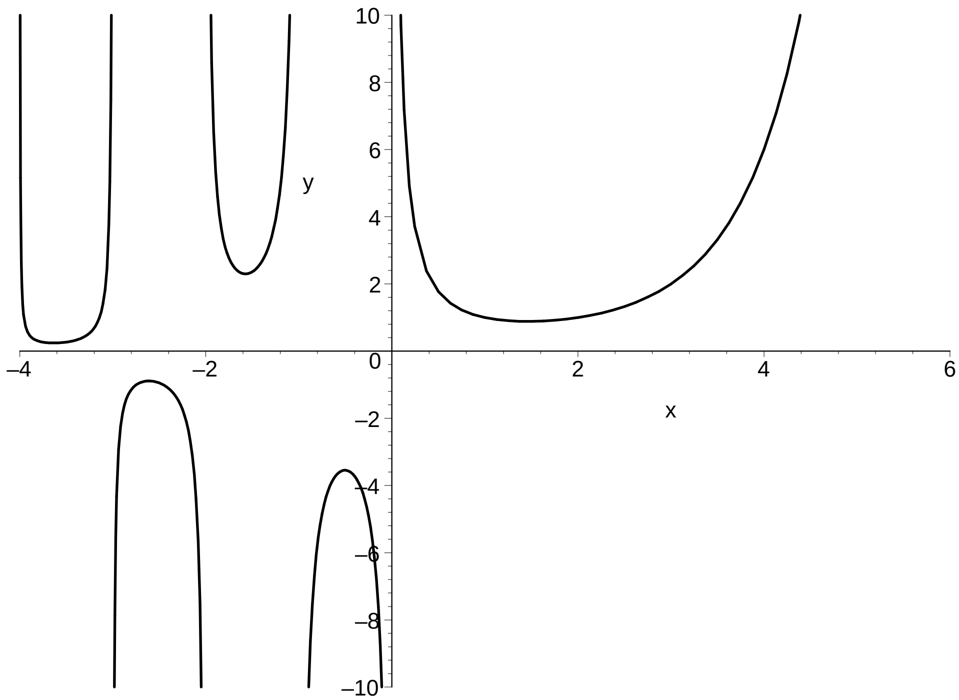

Figure F.1: $\Gamma(x)$, $-4 \leq x \leq 6$