Chapter 4

Complex Numbers, $\mathbb{C}$

As mentioned earlier, the complex numbers arose in studying the roots of equations. We saw that for many early mathematicians the quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the quadratic equation $ax^2 + bx + c = 0$ generated "meaningless" solutions when the discriminant $b^2 - 4ac$ was negative.

When Cardano published his tome *Ars magna* in 1545 he published the more general solutions to the cubic and quartic equations, crediting Tartaglia with the idea for the solution of the cubic and Ferrari with the solution of the quartic.

We saw earlier that these formulæ generated some very odd expressions for even the simplest of solutions that involve the square roots of negative numbers. It was Bombelli who named these quantities imaginary numbers, yet he and others used the rules of arithmetic with these "imaginary" numbers to solve other problems. Euler used complex numbers extensively in number theory in the 1700's and Cauchy developed (discovered) an extensive theory of functions of a complex variable in the 1800's.

It wasn't until mathematicians found a geometric representation as points in the complex plane that these numbers began to become more utilized. This was developed by Wessel in 1799 and Argand in 1806. Gauss used this concept in the 1830's to prove the Fundamental Theorem of Algebra. Augustus F. Möbius, a student of Gauss, used the complex plane and complex functions to classify geometric transformations of the plane. Later Riemann developed a calculus of complex functions from a geometric viewpoint and applied these functions to geometry and number theory.

4.1 The complex plane

The complex numbers that arose in the study of the roots of polynomial equations we of the form $x + y\sqrt{-1}$, where x and y were real numbers. By using the symbolic power of algebra and treating the expressions as if they were binomials with the additional property that $\left(\sqrt{-1}\right)^2 = -1$, they were able to obtain reasonable results. These earlier mathematicians would treat addition as binomial addition:

$$(2 + 4\sqrt{-1}) + (3 - 2\sqrt{-1}) = (2 + 3) + (4 - 2)\sqrt{-1} = 5 + 2\sqrt{-1}$$

and multiplication as:

$$\begin{aligned}(2 + 4\sqrt{-1}) \cdot (3 - 2\sqrt{-1}) &= (2)(3) + (2)(-2)\sqrt{-1} + (4)(3)\sqrt{-1} + (4)(-2)(\sqrt{-1})^2 \\ &= (6 + 8) + (-4 + 12)\sqrt{-1} = 14 - 8\sqrt{-1}'\end{aligned}$$

Through these calculations though, they had no geometric understanding or representation for what $\sqrt{-1}$ might mean.

First, to simplify our notation, let us call $\sqrt{-1} = i$ and i will be treated as an indeterminant with the additional property that $i^2 = -1$.

Given any complex number $a + bi$ we can identify it with the ordered pair (a,b) in the Euclidean plane, $\mathbb{R}^2$. This representation of the complex numbers is called the **complex plane** and it provides a geometric model for the complex numbers.

In this model the point $(0, 1)$ represents our imaginary unit $i = \sqrt{-1}$. It turns out that this is a natural place to represent i. If you think of the plane and any point in the plane, (x, y), then rotation of the plane about the origin by 180° is given by the mapping $(x, y) \mapsto (-x, -y)$. Thus, a rotation by 180° can be thought of geometrically as multiplying by -1. Now, a rotation by 180° is equivalent to two counterclockwise rotations of 90°. Thus, one might view a rotation of 90° as multiplication by a square root of -1, or i.

In this way $i \cdot i$ or i^2 can be pictured by two rotations of 90° about the origin. Three 90° rotations of $(1, 0)$ about the origin sends it to $(0, -1)$ which corresponds to $i^3 = -i$. Four 90° rotations of $(1, 0)$ about the origin maps it back onto itself, which corresponds to multiplication by $i^4 = 1$.

Additionally, if we think of $i = (0, 1)$ then we can view any complex number, $a + bi$, as a linear combination of the unit vectors $(1, 0)$ and $(0, 1)$. This means that we write

$$a + b\sqrt{-1} = a + bi = a \cdot (1, 0) + b \cdot (0, 1),$$

where a and b are real numbers and the multiplication is scalar multiplication.

Once we identify $a+bi$ with the ordered pair (a, b), then we need to find operations on ordered pairs that correspond to what we saw should happen with the complex numbers. These will mimic what we saw above:

Definition 4.1 *A **complex number** is an ordered pair, (a, b), of real numbers a and b with addition and multiplication defined as:*

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2) \tag{4.1}$$

$$(a_1, b_1) \cdot (a_2, b_2) = (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1) \tag{4.2}$$

Now, we can identify the complex number $(x, 0)$ with the real number x because addition and multiplication of numbers of this form give us the usual addition and multiplication of real numbers. By identifying each real number x with the complex number $(x, 0)$ and defining complex addition and multiplication as above, we find that $z = x + iy$ is then the complex number sum of the real number x and the number iy. If $z = x + iy = (x, y)$ the real number x is called the **real part** of z and is denoted by $\mathrm{Re}(z)$, while the real number y is called the **imaginary part** of z and is denoted by $\mathrm{Im}(z)$.

We have seen that the real numbers can be identified as the set of points of the form $(x, 0)$, so the real number line corresponds to the horizontal axis in the complex plane and is usually called the **real axis**. The vertical axis is called the **imaginary axis**.

For the complex number $z = x + iy$, the **complex conjugate** $\bar{z}$ is defined to be $\bar{z} = x - iy$. Note that geometrically this is just reflection through the real axis.

For real numbers, we use the notation $|x|$ to denote the distance from the point x on the real line and the origin. Likewise, we use $|z|$ to denote the distance from z to the origin in the plane. Of course that means that $|z| = \sqrt{x^2 + y^2}$ and is called the **modulus** or **absolute value** of z. Note that $|z|^2 = z \cdot \bar{z}$. (Why?) The set U of complex numbers so that $|z| = 1$ is the unit circle in the plane.

4.2 Movement of Roots in the complex plane

Now, we know (but may need to remind ourselves) what happens as we change the constant term of the quadratic equation. For example look at the following five quadratic equations:

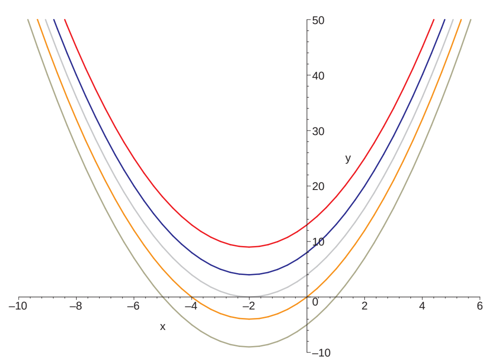
a) $x^2 + 4x - 5$

b) $x^2 + 4x$

c) $x^2 + 4x + 4$

d) $x^2 + 4x + 8$

e) $x^2 + 4x + 13$

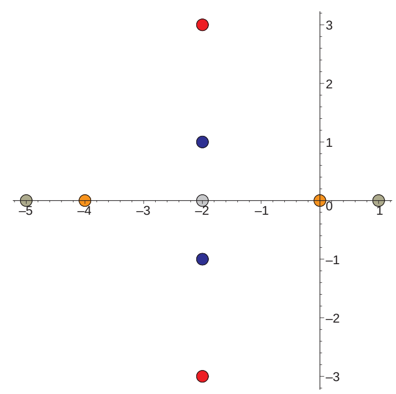


As the constant changes from -5 to 13, the graph of the parabola moves up the y-axis and the parabola goes from intersecting the x-axis twice to just once to no intersections at all. We recall that this means that the real roots go from two to one to none.

But now what happens in the complex plane to the location of the roots. If the roots are real, then the must lie on the real axis. Thus the roots to the first three parabolas all lie on the real axis. From that point on the roots lie above and below point $(-2, 0)$ in conjugate pairs.

We know, in fact, that the roots to the quadratic equation $x^2 + 4x + c = 0$ are

$$S = \left\{ \frac{-4 \pm \sqrt{16 - 4c}}{2} \right\} = \{-2 \pm \sqrt{4 - c}\}.$$



So, when $c < 4$ we have two real roots symmetric on the real axis with respect to -2, and they approach one another as c gets closer to 4. At $c = 4$ the solution set is a single number, -2, and as c increases form 4, the solution set consists of two points symmetric to the real axis on the vertical line $x = -2$. The distance between these points gets larger as c continues to increase.

4.3 Polar Form for Complex Numbers

Since we know that we can describe any point in the plane in terms of a distance from the origin and an angle measured from the positive x-axis, we know that we can do the same for complex numbers. Just as in our usual terminology, we will say that the form $x + iy$ for the complex number z is the **rectangular form** of the complex number.

Recall that the polar form of a point in the plane $z = x + iy$ are the **polar coordinates** $[r, \theta]$, where

$$\begin{aligned} r &= \sqrt{x^2 + y^2}, && (4.3)\\ \cos(\theta) &= \frac{x}{r}, \quad \sin(\theta) = \frac{y}{r}, \text{ and } -\pi < \theta \le \pi. && (4.4) \end{aligned}$$

Note that polar coordinates are not a panacea for problems with rectangular coordinates. First, each point in the plane has an infinitely many polar coordinates. If we know that a complex number, z, has polar coordinates $[r, \theta]$, then z also has $[r, \theta + 2k\pi]$ as polar coordinates for any integer k, as well as $[-r, \theta + (2k + 1)\pi]$ for any integer k.

Note that $r = |z|$. The θ in the second equation is called the **principal argument** for z and is denoted by $\text{Arg}(z)$.

Conversely, if $[r, \theta]$ is any pair of polar coordinates of a point z in the complex plane then $x = r\cos\theta$ and $y = r\sin\theta$ are its rectangular coordinates and $z = x + iy$. This means then that we can express any complex number in the form

$$z = r(\cos\theta + i\sin\theta),$$

and this is called the **polar representation** or **polar form** of z.

4.4 The Geometry of Complex Number Arithmetic

We have seen four different methods of representing complex numbers:

binomial form $a + bi$
rectangular form (a, b)
polar coordinates $[r, \theta]$
polar form $r(\cos\theta + i\sin\theta$

A fifth form, which is quite common, was introduced by Euler. He discovered this through his work with the power series expansions for the exponential function, the sine function and the cosine function. We may have time to look at this later. The exponential form for a complex number is

$$re^{i\theta}.$$

The reasons for looking at these different representations is that oftentimes one property of a number system can be more easily understood in one coordinate system and not in another. We often play these coordinate systems "against each other" to better understand what we are studying.

4.4.1 Addition and Subtraction

It was easy to see when we defined complex numbers that we chose to perform addition "coordinate-wise". We defined:

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2).$$

This corresponds to adding ordered pairs of real numbers as *vectors*. Thus, we can understand addition and subtraction of complex numbers by understanding the addition and subtraction of vectors. Also, we can try to see what properties of complex numbers we might understand better by looking at these operations with this viewpoint. Geometrically, recall that if two vectors are not collinear, then their sum is given by the **Parallelogram Law** of addition. Thus, the same must be true of complex numbers. So, if 0, z_1 and z_2 are not collinear, then $z_1 + z_2$ is the fourth vertex of the parallelogram with consecutive vertices z_1, 0, and z_2. Compare Figure refparallellaw. From our usual consideration in

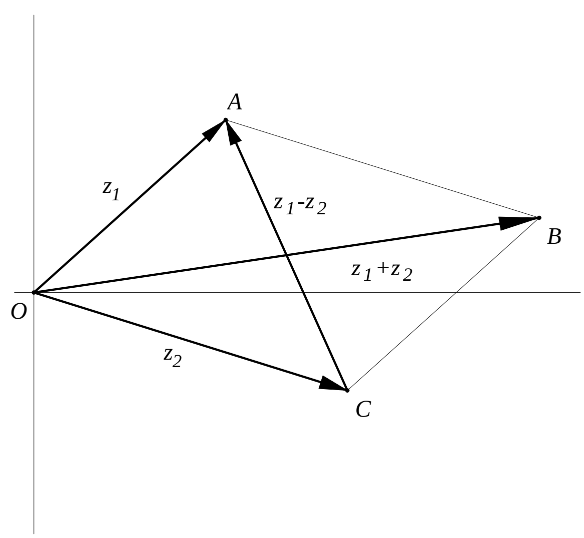


Figure 4.1: Parallelogram Law

vectors, the vector $\vec{AC}$ is the vector that you have to add to z_2 in order to get z_1. Thus, $\vec{AC} = z_1 - z_2$ and its length is the distance from z_1 to z_2.

Lemma 4.1 *The distance between z_1 and z_2 in the complex plane is $|z_1 - z_2|$.*

Note that this is entirely analogous to the situation in the real line. The distance between two real numbers a and b is $|a - b|$. In the reals the equation $|x - a| = b$ is the set of points x which are at a distance b from a. The same will be true in the complex plane. If r is a positive real number, the equation $|z - z_0| = r$ is the set of complex numbers that are at a distance r from z.

Lemma 4.2 (Triangle Inequality) *For all complex numbers z_1 and z_2,*

$$|z_1 + z_2| \le |z_1| + |z_2|. \tag{4.5}$$

Corollary 4.1 *For all complex numbers z_1 and z_2*

$$|z_1 - z_2| \ge ||z_1| - |z_2||. \tag{4.6}$$

4.5 Multiplication and Division in $\mathbb{C}$

We know from the binary form of a complex number how to determine the product of two complex numbers. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then from before we have

$$z_1 z_2 = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1).$$

In particular this gives us that

$$z\bar{z} = x^2 + y^2 = |z|^2,$$

as we noted before.

This identity is what we use to define the quotient of two complex numbers: if $z_2 \neq 0$, then

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{z_2\bar{z}_2} = \frac{z_1\bar{z}_2}{|z_2|^2} = \frac{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}{x_2^2 + y_2^2}.$$

While this is algebraically satisfying and elegant, it does nothing for us geometrically. Neither z_1z_2 nor z_1/z_2 have any immediate geometric meaning — well, at least not while in rectangular form. What happens if we write them in polar form?

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ then

$$\begin{aligned} z_1z_2 &= (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1) \\ &= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 + \cos\theta_2\sin\theta_1)] \\ &= r_1r_2(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)) \end{aligned}$$

Also,

$$\begin{aligned}
\frac{z_1}{z_2} &= \frac{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}{x_2^2 + y_2^2} \\
&= \frac{r_1}{r_2}[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\cos\theta_1\sin\theta_2 - \cos\theta_2\sin\theta_1)] \\
&= \frac{r_1}{r_2}(\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2))
\end{aligned}$$

Thus, to multiply two complex numbers together when given in polar form, multiply their moduli together and add their arguments and to divide them, divide their moduli and subtract their arguments.

Now, if you put them into polar coordinates, it becomes even nicer.

Lemma 4.3 *Let $z_1 = [r_1, \theta_1]$ and $z_2 = [r_2, \theta_2]$. Then*

$$z_1z_2 = [r_1, r_2, \theta_1 + \theta_2]$$

and

$$\frac{z_1}{z_2} = \left[\frac{r_1}{r_2}, \theta_1 - \theta_2\right].$$

Note then that multiplication by a positive real number multiplies the modulus by that real number and does not rotate the complex number at all — rotation through 0 radians. Multiplication by i sends $x + iy$ to $ix - y$ or rotates the complex number through $\pi/2$ radians.

4.6 Powers in $\mathbb{C}$

In polar coordinates we have that $[r, \theta]^2 = [r^2, 2\theta]$. So, by induction we can prove the following.

Lemma 4.4 *For a complex number $z = [r, \theta]$, $z^n = [r, \theta]^n = [r^n, n\theta]$.*

Corollary 4.2 *In polar form if $z = r(\cos\theta + i\sin\theta)$, then*

$$z^n = r^n(\cos(n\theta) + i\sin)n\theta).$$

Definition 4.2 *Suppose that z is a complex number. The set*

$$O(z) = \{z^n \mid n \in \mathbb{N}\}$$

*of all positive integer powers of z is called the **orbit** of z.*

Example 4.1 For $z = i$, we have $i^2 = -1$, $i^3 = -i$ and $i^4 = 1$, thus the orbit of i is the set $O(i) = \{i, -1, -i, 1\}$.

If $|z| > 1$, then the orbit must contain complex numbers with larger and larger moduli. If $|z| < 1$, then the orbit will contain complex numbers with smaller and smaller moduli, approaching 0.

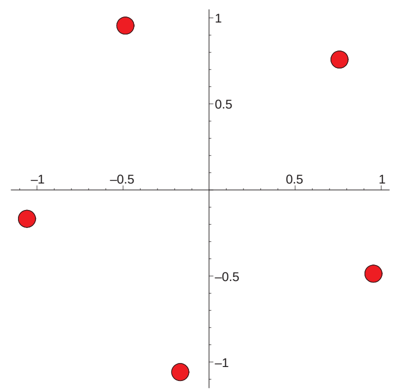
4.7 Roots of complex numbers

If $n \in \mathbb{N}$ is a natural number, then any solution to the equation $x^n = a$ is called an nth root of a. Clearly, if $a = 0$ then 0 is the nth root of 0 for all n — and this is not very interesting.

If $a \neq 0$ then things get more interesting. The existence of real nth roots is a very complicated matter.

1. If n is even and $a > 0$, then a has two real nth roots, one that is positive and is denoted by $\sqrt[n]{a}$ or $a^{1/n}$ and the other negative, denoted by $-\sqrt[n]{a}$ or $-a^{1/n}$.
2. If n is even and $a < 0$, then a has no real nth roots.
3. if n is odd, then a has a unique real nth root, denoted by $\sqrt[n]{a}$.

For complex numbers, however, the situation is much simpler.



Example 4.2 Find all of the fifth roots of $a = -1 - i$.

To do this, first we will rewrite a in polar coordinates:

$$|a| = \sqrt{2} \text{ and } \operatorname{Arg}(a) = -\frac{3\pi}{4}.$$

That means that we need to find all complex numbers $z = [r, \theta]$ so that

$$[r, \theta]^5 = [r^5, 5\theta] = a = [\sqrt{2}, -\frac{3\pi}{4}].$$

Thus, we have that $r^5 = \sqrt{2}$ or $r = \sqrt[10]{2}$ and $5\theta = -\frac{3\pi}{4}$ or $\theta = -\frac{3\pi}{20}$. So,

$$[r, \theta] = \left[\sqrt[10]{2}, -\frac{3\pi}{20}\right].$$

Recall that each complex number has an infinite number of polar coordinates, so we will also look at those representations of a. Those with $r > 0$ are given by $[\sqrt{2}, -\frac{3\pi}{4} + 2k\pi]$, where k is any integer. Note that the modulus does not change, so all of the fifth roots will have first polar coordinate the same: $\sqrt[10]{2}$. The arguments are different so when we solve

$$5\theta = -\frac{3\pi}{4} + 2k\pi,$$

we get

$$\theta = -\frac{3\pi}{20} + \frac{2\pi k}{5}.$$

Therefore each $z_k = [\sqrt[10]{2}, -\frac{3\pi}{20} + \frac{2\pi k}{5}]$ is a fifth root of $a = -1 - i$. Now, the nice part is that sine and cosine are functions that are periodic with period 2π, so we only need worry about $k = 0, 1, 2, 3, 4$ and the five roots are:

$$\left\{ \left[\sqrt[10]{2}, -\frac{3\pi}{20} \right], \left[\sqrt[10]{2}, \frac{\pi}{4} \right], \left[\sqrt[10]{2}, \frac{13\pi}{20} \right], \left[\sqrt[10]{2}, \frac{21\pi}{20} \right], \left[\sqrt[10]{2}, \frac{29\pi}{20} \right] \right\}$$

Notice that when we graph these roots they form the vertices of a regular pentagon.

Theorem 4.1 (DeMoivre's Theorem) *For every natural number $n > 1$, every nonzero complex number z has exactly n distinct complex nth roots:*

$$z_k = \sqrt[n]{|z|} \left(\cos\left(\frac{\operatorname{Arg}(z) + 2\pi k}{n} \right) + i \sin\left(\frac{\operatorname{Arg}(z) + 2\pi k}{n} \right) \right),$$

for $k = 0, 1, \ldots, n-1$.

The points in the complex plane are the vertices of a regular n-gon inscribed in the circle of radius $\sqrt[n]{|z|}$ centered at the origin in the complex plane.

4.7.1 Roots of Unity

For any positive integer n DeMoivre's Theorem shows the real number 1 has n complex nth roots

$$\left[1, \frac{2\pi k}{n} \right], \text{ for } k = 0, 1, dots, n-1.$$

These numbers are usually referred to as the n**th roots of unity**. These solutions to $z^n = 1$ are the vertices of a regular n-gon in the unit circle with 1 at one vertex. In polar form, the first vertex counterclockwise from 1 is the point

$$\omega_n = \cos\left(\frac{2\pi}{n} \right) + i \sin\left(\frac{2\pi}{n} \right).$$

Note that $\omega_n^2 = \cos(\frac{4\pi}{n}) + i \sin(\frac{4\pi}{n})$, $\omega_n^3 = \cos(\frac{6\pi}{n}) + i \sin(\frac{6\pi}{n})$, and in general

$$\omega_n^k = \cos\left(\frac{2\pi k}{n} \right) + i \sin\left(\frac{2\pi k}{n} \right),$$

for any positive integer k. Thus, the set of nth roots of unity can be written as

$$\{1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}\}.$$