

Chapter 5

Functions: How they have changed through History

The National Council of Teachers of Mathematics in *The Principles and Standards for School Mathematics* (2000) states that the secondary school mathematics program must be both broad and deep. (p. 287) They state further that in grades 9-12, students should encounter new classes of functions. Through their high school experiences, they stand to develop deeper understandings of the fundamental mathematical concept of function, (p. 287) Additionally, students need to learn to use a wide range of explicitly and recursively defined functions to model the world around them. Moreover, their understanding of the properties of those functions will give them insights into the phenomena being modeled. (p. 287)

This follows in the heels of a reform in the teaching of calculus at the college level and then at the high school level through Advanced Placement courses. The call from the *Calculus Reform* movement, especially the “Harvard Calculus” group, was to teach the *Rule of Three*. This was an attempt to get students to realize that there were multiple ways to represent and consider functions: numerically, graphically, and analytically. These are not new ways of studying functions. In fact, as we shall see, they are all quite old. The problem though was that in the preceding time, mathematics had been focusing on one particular representation (analytic) and little time was spent on the other representations, even though they were quite useful in areas where the mathematics was applied. The *connections* in mathematics were being missed and ignored.

A central theme of *The Principles and Standards for School Mathematics* is *connections*. There the call is that students develop a much richer understanding of mathematics and its applications when they can view the same phenomena from multiple mathematical perspectives. One way to have students see mathematics in this way is to use instructional materials that are intentionally designed to weave together different content strands. Another means of achieving content integration is to make sure that courses oriented toward any particular content area (such as alge-

bra or geometry) contain many integrative problemsproblems that draw on a variety of aspects of mathematics, that are solvable using a variety of methods, and that students can access in different ways.

In order for us to do this, we need to see how the concept of function arose and how it has changed in the history of mathematics to what we have today.

5.1 History¹

Mathematics is often thought of a being “the study of relations on sets” or “the study of functions on sets” or “the study of dependencies among variable quantities.” We look back at the history of mathematics to see where this concept arose. We must be careful, though, lest we attribute some greater concept to our predecessors than they might have intended.

If you look at Babylonian mathematics you will find tables of squares of the natural numbers, cubes of the natural numbers, and reciprocals of the natural numbers. These tables certainly define functions from \mathbb{N} to \mathbb{N} , but there is no indication that the Babylonians were doing anything other than recording their findings, not looking for any type of relationship between the numbers and their squares or cubes. E. T. Bell suggested in 1945 that to credit the ancient Babylonians with the instinct for the concept of a function by constructing these tables is a the result of modern mathematicians seeing ancient mathematics through modern eyes. Today we may see that the Babylonians were dealing with functions, but there is no indication that they would have thought in these terms.

In the work of Ptolemy we find that he computed chords of a circle which essentially means that he computed trigonometric functions. Suggesting that Ptolemy understood the concept of a function is again begin overly generous. Our eyes may see functions, but not his.

Oresme (1323-1382) was getting closer in 1350 when he described the laws of nature as laws giving a dependence of one quantity on another. regarded as having foreseen and come close to a modern formulation of the concept of function. Oresme developed a geometric theory of latitudes of forms representing different degrees of intensity and extension. In his theory, some general ideas about independent and dependent variable quantities seem to be present.

Galileo was beginning to understand the concept even more clearly. His studies of motion contain the clear understanding of a relation between variables. Again another piece of his mathematics shows how he was beginning to grasp the concept of a mapping between sets. In 1638 he studied the problem of two concentric circles with center O , the larger circle A with diameter twice that of the smaller one B . The familiar formula gives the circumference of A to be twice that of B . But taking any point P on the circle A , then PA cuts circle B in one point. So Galileo had

¹Adapted from [2,3,5]

constructed a function mapping each point of A to a point of B . Similarly if Q is a point on B then OQ cuts circle A in exactly one point. Again he has a function, this time from points of B to points of A . Although the circumference of A is twice the length of the circumference of B they have the same number of points. He also produced the standard one-to-one correspondence between the positive integers and their squares which (in modern terms) gave a bijection between \mathbb{N} and a proper subset.

At almost the same time that Galileo was coming up with these ideas, Descartes (1596-1650) was introducing algebra into geometry in *La Géométrie*. Descartes clearly stated that an equation in two variables, geometrically represented by a curve, indicates a dependence between variable quantities. The idea of derivative came about as a way of finding the tangent to any point of this curve.

It is important to understand that the concept of function developed over time, changing its meaning as well as being defined more precisely as time passed. Early uses of the word *function* did incorporate some of the ideas of the modern concept of function, but in a much more restrictive way.

Newton (1642-1727) was one of the first mathematicians to show how functions could be developed in infinite power series, thus allowing for the intervention of infinite processes. He used *fluent* to designate independent variables, *relata quantitas* to indicate dependent variables, and *genita* to refer to quantities obtained from others using the four fundamental arithmetical operations.

It was Leibniz (1646-1716) who first used the term *function* in 1673. He took function to designate, in very general terms, the dependence of geometrical quantities on the shape of a curve. He also introduced the terms *constant*, *variable*, and *parameter*.

Johann Bernoulli, in a letter to Leibniz written in 1694, described a function as "... a quantity somehow formed from indeterminate and constant quantities. In a paper in 1698 Johann Bernoulli writes of "functions of ordinates." Leibniz wrote to Bernoulli saying, "I am pleased that you use the term function in my sense."

The term function did not appear in a mathematics lexicon published in 1716. Two years later Jean Bernoulli published an article containing his definition of a function of a variable as a quantity that is composed in some way from that variable and constants. Euler (1707-1793), a former student of Bernoulli, later added his touch to this definition speaking of analytical expression instead of quantity. Euler published *Introductio in analysin infinitorum* in 1748 in which he makes the function concept central to his presentation of analysis. Euler defined a function in the book as "A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities." Though Euler gave no definition of what he meant by "analytic expression", we believe that he assumed that the reader would understand it to mean expressions formed from the usual operations of addition, multiplication, powers, roots, *etc.*

At this time the notion of function was then identified in practice with the notion of analytical expression. This representation soon lead to several inconsistencies.

For example, the same function could be represented by several different analytical expressions. This representation also seriously limited the classes of functions that could be considered. In present day terminology, we can say that Euler's definition included just the *analytic functions*. Today we recognize this as a restricted subset of the already small class of continuous functions. Aware of these shortcomings, Euler proposed an alternative definition that did not attract much attention at the time. As far as mainstream mathematics is concerned, the identification of functions with analytical expressions would remain unchanged for all of the 18th century. In the 19th century, however, the notion of function underwent successive enlargements and clarifications that deeply changed its nature and meaning.

Introductio in analysin infinitorum changed the way that mathematicians thought about familiar concepts. Until Euler's work the trigonometric quantities sine, cosine, tangent and others were regarded as lines connected with the circle rather than functions. It was Euler who introduced the functional point of view. The function concept had led Euler to make many important discoveries before he wrote *Introductio in analysin infinitorum*. For example it had led him to define the Gamma function and to solve the problem which had defeated mathematicians for some considerable time, namely summing the series $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \dots$. He showed that the sum was $\pi^2/6$, publishing the result in 1740.

In 1755 Euler published another highly influential book, namely *Institutiones calculi differentialis*. In this book he defined a function in an entirely general way, giving a more modern definition of a function, "If some quantities so depend on other quantities that if the latter are changed the former undergoes change, then the former quantities are called functions of the latter."

This definition applies rather widely and includes all ways in which one quantity could be determined by other. If, therefore, x denotes a variable quantity, then all quantities which depend upon x in any way, or are determined by it, are called functions of x . This might have been a huge breakthrough but after giving this wide definition, Euler then devoted the book to the development of the differential calculus using only analytic functions.

Note that with this and the previous definitions the manner in which you represent the function is tied to the determination of the function. What happens if you can represent the function in two different manners? Is it then a function? The first problems with Euler's definition of types of functions was pointed out in 1780 when it was shown that a mixed function, given by different formulas, could sometimes be given by a single formula. The clearest example of such a function was given by Cauchy in 1844 when he noted that the absolute value function

$$y = |x| = \begin{cases} x & \text{for } x \geq 0 \\ -x & \text{for } x < 0 \end{cases}$$

can be expressed by the single formula $y = \sqrt{x^2}$. Hence dividing functions into one formulation or mixed formulation was meaningless.

However, a more serious objection came through the work of Fourier who stated in 1805 that Euler was wrong. Fourier showed that some discontinuous functions could be represented by what today we call a Fourier series. The distinction between continuous and discontinuous functions, therefore, did not exist. Fourier's work was not immediately accepted and leading mathematicians such as Lagrange did not accept his results at this stage. Fourier's work would lead eventually to the clarification of the function concept when in 1829 Dirichlet proved results concerning the convergence of Fourier series, thus clarifying the distinction between a function and its representation.

Other mathematicians gave their own versions of the definition of a function. Condorcet seems to have been the first to take up Euler's general definition of 1755. In 1778 the first two parts of Condorcet intended five part work *Traité du calcul integral* was sent to the Paris Academy. It was never published but was seen by many leading French mathematicians. In this work Condorcet distinguished three types of functions: explicit functions, implicit functions given only by unsolved equations, and functions which are defined from physical considerations such as being the solution to a differential equation.

Cauchy, in 1821, came up with a definition making the dependence between variables central to the function concept in *Cours d'analyse*. Despite the generality of Cauchy's definition, which was designed to cover the case of explicit and implicit functions, he was still thinking of a function in terms of a formula. In fact he makes the distinction between explicit and implicit functions immediately after giving this definition. He also introduces concepts which indicate that he is still thinking in terms of analytic expressions.

Fourier, in *Théorie analytique de la Chaleur* in 1822, gave a definition which deliberately moved away from analytic expressions. However, despite this, when he begins to prove theorems about expressing an arbitrary function as a Fourier series, he uses the fact that his arbitrary function is continuous in the modern sense!

Dirichlet, in 1837, accepted Fourier's definition of a function and immediately after giving this definition he defined a continuous function (using continuous in the modern sense). Dirichlet also gave an example of a function defined on the interval $[0, 1]$ which is discontinuous at every point, namely

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

Around this time many pathological functions were constructed. Cauchy gave an early example when he noted that

$$f(x) = \begin{cases} e^{-1/x^2} & \text{for } x \neq 0, \\ 0 & x = 0, \end{cases}$$

is a continuous function which has all its derivatives at 0 equal to 0. It therefore has a Taylor series which converges everywhere but only equals the function at 0. In 1876

Paul du Bois-Reymond made the distinction between a function and its representation even clearer when he constructed a continuous function whose Fourier series diverges at a point. This line was taken further in 1885 when Weierstrass showed that any continuous function is the limit of a uniformly convergent sequence of polynomials. Earlier, in 1872, Weierstrass had sent a paper to the Berlin Academy of Science giving an example of a continuous function which is nowhere differentiable.

Goursat, in 1923, gave the definition which appears in most textbooks today: *One says that y is a function of x if to a value of x corresponds a value of y . One indicates this correspondence by the equation $y = f(x)$.*

5.2 Timeline

Here is a non-exhaustive timeline of some of the individuals mentioned above from [2].

Apollonius (c. 262 190 BCE) wrote the *Conic Sections* which was a thorough geometric study of the properties of the curves that we call conic sections.

Diophantus (c. 250) wrote *Arithmetica* in which he took some of the first known steps to move from a verbal algebra towards a symbolic algebra.

Pappus (c. 300 350) wrote the *Synagoge*, also called *The Collection*. This book contained a compilation of some of Apollonius theorems which both Fermat and Descartes addressed algebraically in the 1630s.

Omar Khayyam (1048 1131) used his detailed knowledge of conic sections to solve cubic equations by finding intersection points of certain of these curves.

Sharaf al-Din al-Tusi (d. 1213) treated cubic equations by finding maximum and minimum values of the related “functions.”

Nicole Oresme (c. 1320 1382) wrote *Tractatus de configurationibus qualitatum et motum* around 1350, giving a geometric proof of the Merton mean speed theorem.

Francois Vite (1540 1603) published *In artem analyticam isagoge* in 1591, which contained the first systematic use of letters for both variables and coefficients.

Galileo Galilei (1564 1642) in 1604 began his investigations of falling bodies and was one of the first to apply mathematics to the study of motion.

Ren Descartes (1596 1650) published *Discours de la methode*, including as an essay *La geometrie*, widely considered the co-beginning (with Fermat's *Isogoge*) of analytic geometry. *La geometrie* also drew heavily from Vite's symbolism and extended it.

Pierre de Fermat (1601–1665) wrote *Ad locus planos et solidos isagoge* in 1637 (published in 1679), generally considered the co-beginning (with Descartes *La geometrie*) of analytic geometry.

Isaac Newton (1642–1727) clearly considered the concept, if not the term, of function in some of his earliest work on the calculus.

Gottfried Wilhelm Leibniz (1646–1716) in 1673 first used the word “function” in a sense close to its modern meaning.

Johann Bernoulli (1667–1748) introduced the concept of function into new areas.

Leonhard Euler (1707–1783) exerted a major influence on notation and the concept of functions; in 1748 he published *Introductio in analysin infinitorum* in which he stated “mathematics is a science of functions.”

Jean Baptiste Joseph Fourier (1768–1830) demonstrated that a very wide variety of functions could be represented by an infinite trigonometric series.

Gustav Peter Lejeune-Dirichlet (1805–1859) continued to push the concept of function into new and more technically rigorous areas.

5.3 A Catalogue of Definitions²

Isaac Newton 1713

I call any quantity a genitum which is generated or produced in arithmetic by the multiplication, division, or extraction of the root of any terms whatsoever—These quantities I here consider as variable and indetermined, and increasing or decreasing, as it were, by a continual motion or flux.

Johann Bernoulli 1718

I call a function of a variable magnitude a quantity composed in any manner whatsoever from this variable magnitude and from constants.

Leonhard Euler 1748

A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities... If, therefore, x denotes a variable quantity, then all quantities which depend upon x in any way or are determined by it are called functions of it.

²Adapted from [2]

Leonhard Euler 1755

If some quantities so depend on other quantities that if the latter are changed the former undergo change, then the former quantities are called functions of the latter.

Joseph-Louis Lagrange 1797

We define a function of one or more quantities any mathematical expression in which those quantities appear in any manner, linked or not with some other quantities that are regarded as having given and constant values, whereas the quantities of the function may take all possible values.

Jean Baptiste Joseph Fourier 1822

In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x , there is an equal number of ordinates $f(x)$. We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity.

Nikolai Ivanovich Lobachevsky 1834

General conception demands that a function of x be called a number which is given for each x and which changes gradually together with x . The value of the function could be given by an analytical expression, or by a condition which offers a means for testing all numbers and selecting one of them; or, lastly, the dependence may exist but remain unknown.

Karl Weierstrass 1861

Two variable magnitudes may be related in such a way that to every definite value of one there corresponds a definite value of the other; then the latter is called a function of the former.

Hermann Hankel 1870

y is called a function of x when to every value of the variable quantity x inside of a certain interval there corresponds a definite value of y , no matter whether y depends on x according to the same law in the entire interval or not, or whether the dependence can be expressed by a mathematical operation or not.

Nicolas Bourbaki 1939

Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all x an element of E , there exists a unique y an element of F which is in the given relation with x .

5.4 Modern Definitions

As you can see, the concept of function has been a long time in the making. The definition of functions that we give in our high school classrooms have been through many, many changes. We are offering the students the final product of centuries of thought. We offer a mathematically exact, precise definition. The use of other representations can (and will) be used to try to understand what the definitions mean and why we have chosen these definitions.

From what we have discussed above, the idea of a function is to express a relationship between the elements of two sets. If A and B are sets, then a *function from A to B* is often described as a *rule* or *process* that associates each element of A with one and only one element of B .

Definition 5.1 *A function is a rule that assigns to each element of a set A a unique element of a set B , where B may or may not equal A .*

The set A is called the **domain** of the function f , the set B the **codomain**, and the subset of B consisting of those elements that are images under the function f of some element of its domain is called the **range** of the function f . If f associates a in A to b in B , then the element b is called the **image of a under f** or **the value of f at a** , and a is called the **preimage of b under f** .

There are a number of notations for functions in use in mathematics. The common notations are when f associates a with b , then the **functional notation** or $f(x)$ notation is written **$f(a)=b$** . The **arrow** or **mapping** notation is written $f: a \rightarrow b$.

One positive aspect of the arrow notation is that it conveys the idea that there is an action that associates the elements from A to the corresponding elements of B . This can be written as $f: A \rightarrow B$ only to indicate the domain and codomain, in which case the notation for elements is $f: a \mapsto b$. When the arrow notation is used, we will say that the function f **maps** the element a to b and we call f a **mapping** or **map** from A to B . We say that f maps A **onto** B if every element of B is in the range; i.e., $f(A) = B$.

The value in the domain of a function is called an **argument** of f . Then, the variable that we use to stand for the argument is called the **independent variable**. The variable that stands for the value of the function f is called the **dependent variable**. These are also referred to as *input* and *output* variables.

Example 5.1 The rule that assigns to each number the square of that number. Here we can express the function as a formula, either $y = x^2$ or $f(x) = x^2$.

For the function $f: x \rightarrow y$ many authors consistently use the single letter f to name the function and distinguish this from the symbol $f(x)$ used to identify the values of the function. But more broadly in mathematics this distinction is not made and $f(x)$ may stand for a function and also its values. Using the symbol $f(x)$ to stand for a function allows the independent variable to be explicitly identified.

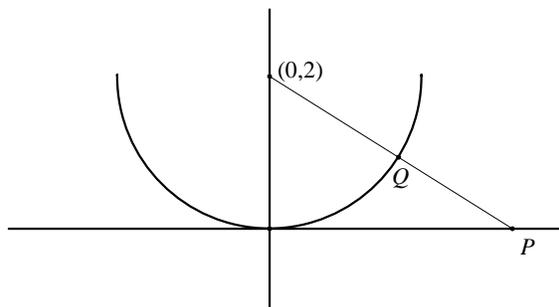
Functions are not always expressed in terms of formulæ. The relationship may be expressed by a table listing all of its values or by a graph.

For example, the population given by the U.S. Bureau of the Census for the nation, the state, or a county is a function from the set of years to the set of natural numbers.

	US	NC	Meck	Wake
2005	296,410,404	8,411,041	796,372	748,815
2000	281,421,906	8,049,313	695,454	627,846
1990	248,709,873	6,628,637	511,433	423,380
1980	226,545,805	5,881,766	404,270	301,327
1970	203,211,926	5,082,059	354,656	228,453
1960	179,323,175	4,556,155	272,111	169,082
1950	151,325,798	4,061,929	197,052	136,450
1940	132,164,569	3,571,623	151,826	109,544
1930	123,202,624	3,170,276	127,971	94,757
1920	106,021,537	2,559,123	80,695	75,155
1910	92,228,496	2,206,287	67,031	63,229
1900	76,212,168	1,893,810	55,268	54,626

Example 5.2 Look at the following example. This example is easier to describe geometrically than analytically. We can, however, use our knowledge of analytic geometry to derive an analytic formula for the function.

The function is a mapping from the real line to the open interval $(-\pi, \pi)$ as follows. Take the circle of radius 2 centered at the point $(0, 2)$ in the plane. We can identify the interval $(-\pi, \pi)$ with the lower semicircle of this circle — the arc through $(-2, 2)$, $(0, 0)$, and $(2, 2)$, not including $(-2, 2)$ and $(2, 2)$. That arc has length $2\pi r/2 = 2\pi$. Draw a line from $(0, 2)$ to any point, P , on the x -axis. That line will intersect the lower half of the circle in exactly one point, Q . We then map Q to the length of the arc OQ — positive if it is in the first quadrant, negative if Q lies in the second quadrant. Then the mapping is a function from \mathbb{R} to $(-\pi, \pi)$. Note that it is the open interval since the line through the points $(2, 2)$, $(0, 2)$, and $(-2, 2)$ is parallel to the x -axis and, thus, does not intersect it.



The function is represented in Figure 5.4.

Using what we know about analytic geometry we can probably write down a formula for f . If $P = (a, 0)$, then the line from $(0, 2)$ to P has slope $-2/a$. It has

y -intercept 2, so the equation of the line is $y = -\frac{2}{a}x + 2$. Where does this line intersect the circle?

The equation of the circle is $x^2 + (y - 2)^2 = 4$. The lower half of the circle is given by $y = 2 - \sqrt{4 - x^2}$. We need to find the point of intersection of this line and this curve. So setting $-\frac{2}{a}x + 2 = 2 - \sqrt{4 - x^2}$ gives us $x = 2a/\sqrt{4 + a^2}$. This then makes $y = 2 - 4/\sqrt{4 + a^2}$. So the point on the circle is $(2a/\sqrt{4 + a^2}, 2 - 4/\sqrt{4 + a^2})$. Now, we need to find the arclength from $(0, 0)$ to this point. We know that the arclength is 2θ where θ is the central angle. Looking at the angle, we see that the x coordinate gives us one leg of the defining triangle for θ and the other leg is given by $2 - y$. This means that the tangent of θ is given by:

$$\tan \theta = \frac{x}{2 - y} = \frac{\frac{2a}{\sqrt{4+a^2}}}{2 - (2 - \frac{4}{\sqrt{4+a^2}})} = \frac{a}{2}.$$

Thus, we map $P = (a, 0)$ to $\arctan\left(\frac{a}{2}\right)$.

There might be a “rule” to describe this function, but it might not be easily discovered or written. In the case of a correspondence where the idea of a function as a rule we need a better definition. This is done in the language of sets.

Definition 5.2 For any sets A and B a function f from A to B , $f: A \rightarrow B$ is a subset f of the Cartesian product $A \times B$ such that every $a \in A$ appears once and only once as the first element of an ordered pair $(a, b) \in f$.

This characterization of function now allows us to associate a *graph* with a function. Notice also that this is a very precise definition, but it is removed from the concept of a function “doing something”. This is a more static definition and does not give us the feeling that the function is moving or mapping. Notice, though, that this definition is independent of any *a priori* knowledge of the sets A and B . This is the most general definition of function, and so is easily generalized to other settings.

5.5 Properties of Functions

Definition 5.3 A function, $f: A \rightarrow B$ is a one-to-one function if and only if every element $b \in B$ is the image of at most one element $a \in A$. Symbolically, this can be expressed as f is one-to-one if and only if for all $x_1, x_2 \in A$, $f(x_1) = f(x_2)$ implies $x_1 = x_2$.

Definition 5.4 If $f = \{(x, y) \mid y = f(x)\}$ and f is one-to-one, then the function $\{(y, x) \mid (x, y) \in f\}$ is called the inverse of f and denoted by f^{-1} .

Lemma 5.1 If $f: A \rightarrow B$ is a one-to-one function with range $f(A)$, then

$$f^{-1} = \{(y, x) \in f(A) \times A \mid (x, y) \in f\}$$

is a one-to-one function with domain $f(A)$ and range A .

Definition 5.5 If $f: A \rightarrow B$ and $g: B \rightarrow C$ are functions, then the composite function $(g \circ f): A \rightarrow C$ is the subset $g \circ f \subset A \times C$ defined as follows:

$$g \circ f = \{(a, g(f(a))) \in A \times C \mid a \in A\}.$$

If $f: X \rightarrow Y$ and $U \subset X$ is a subset of X , then the set

$$f_U = \{(x, y) \mid x \in U\}$$

is a function from U to Y called the *restriction of f to U* . The restriction $f_U: U \rightarrow Y$ has the equation

$$f_U(x) = f(x) \text{ for all } x \in U.$$

For any set C the symbol I_C denotes the identity function on C given by

$$I_C = \{(x, x) \mid x \in C\}.$$

Lemma 5.2 Suppose $f: A \rightarrow B$ is a given function. Then there is a function $g: B \rightarrow A$ such that

$$g \circ f = I_A \text{ and } f \circ g = I_B$$

if and only if f is a one-to-one function and $g = f^{-1}$.

5.5.1 Monotone Functions

A function which for all values of x in some interval of the real line has the same value $f(x) = a$ is called a *constant* function. A function $f(x)$ for which an increase in the value of x causes an increase in the value of the function, *i.e.* $f(x_1) < f(x_2)$ whenever $x_1 < x_2$ is called a *monotonically increasing*, or *strictly increasing* function. On the other hand, if whenever we increase the value of x we decrease the functional value, the function is called *monotonically decreasing*. Note that a monotone function always maps different values of x to different values of $f(x)$ so that any monotone function is one-to-one.

As an aside here, for some authors there are *increasing* functions and *monotonically increasing* functions and they are different. You must check the definitions when working here. The following definition is called *increasing, nondecreasing, monotonically nondecreasing*: $f(x_1) \geq f(x_2)$ whenever $x_1 > x_2$. Note then that the constant function satisfies this condition. We might want to say that it is not decreasing, so it is a nondecreasing function, but *increasing*? There are different reasons for including this definition in the given problem.

5.5.2 Even and Odd Functions

A function $f(x)$ is an *even* function if $f(-x) = f(x)$ for all x . A function $f(x)$ is an *odd* function if $f(-x) = -f(x)$ for all x . A function may be even or odd or neither. Examples are $f(x) = x^2$, $f(x) = x^3$ and $f(x) = 2x + 1$.

Graphically, where even and odd arise is from looking at symmetries of the graph. If we replace a by $-a$ in the argument, then we are looking at what happens across the y -axis. Thus, an even function is symmetric with respect to the y -axis because we get the same value when replacing a by $-a$; $f(-a) = f(a)$. If you can draw its graph for $x > 0$, then you need only reflect that across the y -axis to obtain the other half of its graph.

If $f(x)$ is an odd function, then we don't have that property, but instead we have that the value of $f(-a)$ is the opposite of the value of $f(a)$. In other words, we would reflect the graph across the y -axis and then across the x -axis. This is known as reflection through the origin, because $(a, f(a))$ and $(-a, -f(a))$ lie on the same line through the origin.

5.6 Bibliography for Chapter 5

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