Chapter 10

Continuity

In *Introductio in analysin infinitorum* Euler introduced continuous, discontinuous and mixed functions. The first two of these concepts, however, have different modern meanings. An Euler continuous function was one which was expressed by a single analytic expression, a mixed function was expressed in terms of two or more analytic expressions, and an Euler discontinuous function included mixed functions but was a more general concept. Euler did not clearly indicate what he meant by an Euler discontinuous function although it was clear that he thought of them as more general than mixed functions. He later defined them as those functions which had arbitrarily hand-drawn curves as their graphs.

In 1821 Cauchy gave a definition making the dependence between variables central to the function concept. Despite the generality of his definition, which was designed to cover both explicit and implicit functions, Cauchy thought of a function in terms of a formula.

Fourier, in *Théorie analytique de la Chaleur* in 1822, gave a definition which deliberately moved away from analytic expressions. Dirichlet, in 1837, accepted Fourier's definition of a function and immediately after giving this definition he defined a continuous function (using continuous in the modern sense). Dirichlet also gave an example of a function defined on the interval $[0, 1]$ which is discontinuous at every point, namely $f(x)$ which is defined to be 0 if x is rational and 1 if x is irrational.

In 1838 Lobachevsky gave a definition of a general function which still required it to be continuous.

> *A function of x is a number which is given for each x and which changes gradually together with x. The value of the function could be given either by an analytic expression or by a condition which offers a means for testing all numbers and selecting one from them, or lastly the dependence may exist but remain unknown.*

Mathematicians around this time began to construct many pathological functions.

Cauchy gave an early example when he noted that

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ f(0) = 0, \end{cases}$$

is a continuous function satisfying

$$f^{(n)}(0) = 0.$$

Therefore, it has a Taylor series which converges everywhere but only equals the function at 0. In 1876 Paul du Bois-Reymond made the distinction between a function and its representation even clearer when he constructed a continuous function whose Fourier series diverges at a point. This was taken further in 1885 when Weierstrass showed that any continuous function is the limit of a uniformly convergent sequence of polynomials. Earlier, in 1872, Weierstrass had sent a paper to the Berlin Academy of Science giving an example of a continuous function which is nowhere differentiable.

10.1 Continuous Functions

For us the important issues of function are:

1. the set on which f is defined, called the *domain of* f and written $\mathrm{dom}(f)$;
2. the assignment, rule, or formula specifying the value $f(x)$ of f at each $x \in \mathrm{dom}(f)$.

Our interest lies in functions whose domain is a subset of the reals and whose values lie in $\mathbb{R}$. These are called *real-valued functions of a real variable.* A subtle difference, that is a very important difference, is that the symbol f represents the function, itself, while $f(x)$ represents the *value* of the function at x. We normally give a function by specifying its values and without mentioning its domain. In this case the domain is understood to be the *natural domain*: the largest subset of $\mathbb{R}$ on which the function is a well defined real-valued function. Therefore when we talk about "the function $f(x) = 1/x$" is shorthand for

> "the function f which sends x to $1/x$ with natural domain $\{x \in \mathbb{R} \mid x \neq 0\}$."

Similarly, the natural domain of $g(x) = \sqrt{9 - x^2}$ is $[-3, 3]$ and the natural domain of $\sec x$ is the set of real numbers x not of the form $(2n + 1)\pi/2$, $n \in \mathbb{Z}$.

10.1.1 Definition of Continuity

Definition 10.1 *Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. The function f is* continuous at $x = a$ *if, for every sequence of real numbers $\{x_n\} \subset$* $\mathrm{dom}(f)$ *that converges to a, we have that*

$$\lim_{n\to\infty} f(x_n) = f(a).$$

If f is continuous at each point of a set $A \subset \mathrm{dom}(f)$, then we say that f is continuous on S. *The function f is said to be continuous if it is* continuous on $\mathrm{dom}(f)$.

Note that this means that we will say that the function $f(x) = 1/x$ is continuous or that the function $g(x) = \sqrt{1-x^2}$ is continuous. This is a choice, but we will stick with this choice through the remainder of the course.

This definition implies that the values of $f(x)$ are close to the value $f(a)$ when the values of x are close to a. The first theorem shows that this definition of continuity corresponds to the usual definition on the real line.

Theorem 10.1 *Let f be a real-valued function whose domain is a subset of $\mathbb{R}$. Then f is continuous at $a \in \mathrm{dom}(f)$ if and only if*

> *for each $\epsilon > 0$ there exists $\delta > 0$ so that if $x \in \mathrm{dom}(f)$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$.*

Proof: Suppose that this condition holds and let $\{x_n\} \subset \mathrm{dom}(f)$ which converges to a. We need to show that $\lim_{n\to\infty} f(x_n) = f(a)$. Let $\epsilon > 0$ be given. By the condition there exists a $\delta > 0$ such that if $x \in \mathrm{dom}(f)$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$. Since $\lim x_n = a$, there is a natural number N so that if $n > N$ then $|x_n - a| < \delta$. Therefore, if $n > N$ then we have that $|f(x_n) - f(a)| < \epsilon$. Thus, $\lim_{n\to\infty} f(x_n) = f(a)$.

Now assume that f is continuous at $x = a$. We will use proof by contradiction to finish the proof of this theorem. Thus, we will assume that the condition given in the theorem does not hold. That is, we will assume that it is not true that for each $\epsilon > 0$ there exists $\delta > 0$ so that if $x \in \mathrm{dom}(f)$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$.

Thus, it must be true that there exists an $\epsilon > 0$ so that the condition

"if $x \in \mathrm{dom}(f)$ and $|x-a| < \delta$ then $|f(x) - f(a)| < \epsilon$"

fails for each $\delta > 0$. In particular, the condition

"if $x \in \mathrm{dom}(f)$ and $|x-a| < \frac{1}{n}$ then $|f(x) - f(a)| < \epsilon$"

fails for each $n \in \mathbb{N}$. So for each $n \in \mathbb{N}$ there exists $x_n \in \mathrm{dom}(f)$ such that $|x-a| < \frac{1}{n}$ and $|f(x) - f(a)| \geq \epsilon$. Therefore we have that $\lim_{n\to\infty} x_n = a$ but it is not possible for $\lim_{n\to\infty} f(x_n) = f(a)$ since $|f(x) - f(a)| \geq \epsilon$ for all n. This implies that f cannot be continuous at a, contrary to our assumption. ∎

Why did we define continuity in terms of sequences? Many times it is easier to work with this definition than the usual ϵ-δ definition.

Example 10.1 Let $f(x) = 3x^2 + 2x - 1$ for $x \in \mathbb{R}$. Prove that f is continuous on $\mathbb{R}$ by

1. using the definition,
2. using the ϵ-δ definition.

First, from the definition, suppose that $\lim_{n\to\infty} x_n = a$. Then

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} (3x_n^2 + 2x_n - 1) = 3\left(\lim_{n\to\infty} x_n\right)^2 + 2\left(\lim_{n\to\infty} x_n\right) - 1 = 3a^2 + 2a - 1 = f(a)$$

by using all of the limit theorems from Chapter 6. Hence f is continuous on $\mathbb{R}$.

Now, we want to use the ϵ-δ of continuity to show that f is continuous on $\mathbb{R}$. Let $a \in \mathbb{R}$ and let $\epsilon > 0$. We need to show $|f(x) - f(a)| < \epsilon$ provided that $|x - a|$ is sufficiently small. Note,

$$\begin{aligned} |f(x) - f(a)| &= |(3x^2 + 2x - 1) - (3a^2 + 2a - 1)| = |(3x^2 - 3a^2) - (2x - 2a)| \\ &= 3|x - a| \cdot |x + a| + 2|x - a| \end{aligned}$$

We need a bound on $|x + a|$ that does ***not*** depend on x. Note that if $|x - a| < 1$, then $|x| < |a| + 1$ and hence $|x + a| \leq |x| + |a| < 2|a| + 1$. Thus,

$$|f(x) - f(a)| \leq 3|x - a| \cdot (2|a| + 1) + 2|x - a|$$

provided that $|x - a| < 1$. To guarantee that $3|x - a| \cdot (2|a| + 1) + 2|x - a| < \epsilon$, we want each piece less than $\epsilon/2$. For the second part, we would want $|x - a| < \epsilon/4$, but the first $|x - a|$ needs to be less than $\frac{\epsilon}{6(2|a|+1)}$ and less than 1. So, put

$$\delta = \min\left\{1, \frac{\epsilon}{6(2|a| + 1)}\right\}.$$

Then clearly $|x - a| < \epsilon/6 < \epsilon/4$. Thus, by taking $|x - a| < \delta$ we have that $|f(x) - f(a)| < \epsilon$, as needed.

Example 10.2 Show that

$$f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$$

is continuous at $x = 0$.

Let $\epsilon > 0$. Now, $|f(x) - f(0)| = |f(x)|^2 \leq x^2$ for all x. Since we need this to be less than ϵ¡ then put $\delta = \sqrt{\epsilon}$. Then $|x - 0| < \delta$ implies $x^2 < \delta^2 = \epsilon$, so f is continuous at $x = 0$.

We should note that $\sin(\frac{1}{x})$ and $\frac{1}{x}\sin(\frac{1}{x})$ letting both be 0 at $x = 0$ are not continuous at $x = 0$. Essentially, the reason is that the first tries to approach all values between -1 and 1 as $x \to 0$ while in the second example the function tries to take on all real values as $x \to 0$.

Let f be a real valued function. For $k \in \mathbb{R}$, kf is the function defined by $(kf)(x) = k \cdot f(x)$ for $x \in \mathrm{dom}(f)$. Likewise we denote by $|f|$ the function defined by $|f|(x) = |f(x)|$ for $x \in \mathrm{dom}(f)$.

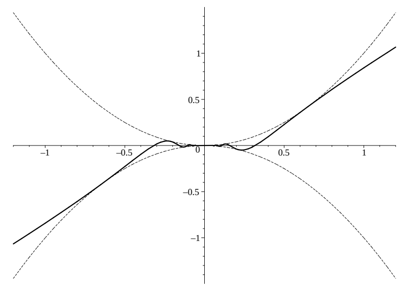

Figure 10.1: $y = x^2 \sin\frac{1}{x}$

Theorem 10.2 *Let $f\colon \mathbb{R} \to \mathbb{R}$. If f is continuous at $a \in \mathrm{dom}(f)$ then kf and $|f|$ are continuous at $x = a$.*

PROOF: Let $x_n \to a$ with $\{x_n\} \subset \mathrm{dom}(f)$. Since f is continuous at $x = a$ we have that $\lim_{n\to\infty} f(x_n) = f(a)$, then from Chapter 6 we have that $\lim_{n\to\infty} kf(x_n) = kf(a)$ which proves that kf is continuous at $x = a$.

To show that $|f|$ is continuous at $x = a$ we need to know that $\lim_{n\to\infty} |f(x_n)| = |f(a)|$. This follows from one of the homework problems and the inequality

$$||f(x_n)| - |f(a)|| \le |f(x_n) - f(a)|.$$

∎

Remember that if we have two functions, there are numerous ways to combine them to get new functions:

$$\begin{aligned}
(f \pm g)(x) &= f(x) \pm g(x) & fg(x) &= f(x)g(x) \\
(f/g)(x) &= \frac{f(x)}{g(x)} & g \circ f(x) &= g(f(x)) \\
\max(f,g)(x) &= \max\{f(x), g(x)\} & \min(f,g)(x) &= \min\{f(x), g(x)\}
\end{aligned}$$

The domains of fg, $f \pm g$, $\max(f,g)$, and $\min(f,g)$ are $\mathrm{dom}(f) \cap \mathrm{dom}(g)$, the domain of f/g is $\mathrm{dom}(f) \cap \{x \in \mathrm{dom}(g) \mid f(x) \neq 0\}$, and the domain of $g \circ f$ is $\{x \in \mathrm{dom}(f) \mid f(x) \in \mathrm{dom}(g)\}$.

Theorem 10.3 *Let $f, g\colon \mathbb{R} \to \mathbb{R}$ be continuous at $a \in \mathbb{R}$. Then*

i) $f \pm g$ is continuous at $x = a$;

ii) fg is continuous at $x = a$; and

iii) f/g is continuous at $x = a$ if $g(a) \neq 0$.

This follows from the similar results about the limits of sequences.

Theorem 10.4 *If f is continuous at $x = a$ and g is continuous at $x = f(a)$, then the composite function $g \circ f(x)$ is continuous at $x = a$.*

PROOF: We are told that $a \in \text{dom}(f)$ and $f(a) \in \text{dom}(g)$. Let $\{x_n\}$ be a sequence in $\text{dom}(f)$ that converges to a. Since f is continuous at a, then we have that

$$\lim_{n\to\infty} f(x_n) = f(a).$$

Since the sequence $\{f(x_n)\}$ converges to $f(a)$ and since g is continuous at $f(a)$, we have that

$$\lim_{n\to\infty} g(f(x_n)) = g(f(a)).$$

Hence, $g \circ f$ is continuous at $x = a$. ∎

To show that $\max(f, g)$ is continuous at $x = a$, we only need to note that

$$\max(f, g) = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|.$$

For $\min(f, g)$ one only need to note that $\min(f, g) = -\max(-f, -g) = \frac{1}{2}(f + g) - \frac{1}{2}|f - g|$.

10.2 Properties of Continuous Functions

Let's review a few important facts about sequences.

Lemma 10.1 *If $x_n \in [a, b]$ for all $n \in \mathbb{N}$ and $\{x_n\}$ converges to x. Then $x \in [a, b]$.*

PROOF: First we will show $a \leq x$. Suppose $x < a$. Choose $\epsilon > 0$ with $\epsilon < a - x$. Then there is no element of the sequence that is in the interval $(x - \epsilon, x + \epsilon)$, which is a contradiction. We us a similar argument to show that $b \geq x$. ∎

Theorem 10.5 (Nested Interval Theorem) *Suppose $I_n = [a_n, b_n]$ where $a_n < b_n$ for $n \in \mathbb{N}$ and $I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots$. Then*

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

PROOF: Note that we have

$$a_1 \leq a_2 \leq a_3 \cdots \leq a_n \leq \cdots \leq b_n \leq \cdots \leq b_2 \leq b_1.$$

Then each b_i is an upper bound for the set $A = \{a_1, a_2, \ldots\}$. By the Least Upper Bound Axiom, we can find a least upper bound for A, call it α.

We claim that $\alpha \in I_n$ for all $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Since α is an upper bound for A, $a_n \leq \alpha$. But b_n is also an upper bound for A and α being the least upper bound implies that $\alpha \leq b_n$. Hence $\alpha \in I_n$ for all $n \in \mathbb{N}$ and $\alpha \in \bigcap_{n=1}^{\infty} I_n$. ∎

Theorem 10.6 (Bolzano-Weierstrass Theorem) *Every bounded sequence has a convergent subsequence.*

Proof: Let $\{x_i\}$ be a bounded sequence. There is $M \in \mathbb{R}$ such that $|x_i| \leq M$ for all $i \in \mathbb{N}$. We will construct inductively a sequence of intervals

$$I_0 \supset I_1 \supset I_2 \supset \dots$$

such that

1. I_n is a closed interval $[a_n, b_n]$ where $b_n - a_n = \dfrac{2M}{2n}$;
2. $\{i \mid x_i \in I_n\}$ is infinite.

We let $I_0 = [-M, M]$. This closed interval has length $2M$ and $x_i \in I_0$ for all $i \in \mathbb{N}$.

Suppose we have $I_n = [a_n, b_n]$ satisfying 1 and 2. Let c_n be the midpoint $c_n = \dfrac{a_n + b_n}{2}$. Each of the intervals $[a_n, c_n]$ and $[c_n, b_n]$ is half the length of I_n. Thus they both have length

$$\frac{1}{2}\frac{2M}{2^n} = \frac{2M}{2^{n+1}}.$$

If $x_i \in I_n$, then $x_i \in [a_n, c_n]$ or $x_i \in [c_n, b_n]$, or, possibly, both. Thus at least one of the sets

$$\{i \mid x_i \in [a_n, c_n]\} \text{ and } \{i \mid x_i \in [c_n, b_n]\}$$

is infinite. If the first is infinite, we let $a_{n+1} = a_n$ and $b_{n+1} = c_n$. If the second is infinite, we let $a_{n+1} = c_n$ and $b_{n+1} = b_n$. Let $I_{n+1} = [a_{n+1}, b_{n+1}]$. Then 1 and 2 are satisfied.

By the Nested Interval Theorem, there is $\alpha \in \bigcap_{n=1}^{\infty} I_n$. We need to find a subsequence converging to α.

Choose $i_1 \in \mathbb{N}$ such that $x_{i_1} \in I_1$. Suppose we have i_n. We know that $\{i \mid x_i \in I_{n+1}\}$ is infinite. Therefore we can find $i_{n+1} > i_n$ such that $x_{i_{n+1}} \in I_{n+1}$. This allows us to construct a sequence of natural numbers $i_1 < i_2 < i_3 < \dots$ where $i_n \in I_n$ for all $n \in \mathbb{N}$. We finish the proof by showing that the subsequence $\{x_{i_n}\} \to \alpha$.

Let $\epsilon > 0$. Choose N such that $\epsilon > \frac{2M}{2^N}$. Suppose $n \geq N$. Then $x_{i_n} \in I_n$ and $\alpha \in I_n$. Thus

$$|x_{i_n} - \alpha| \leq \frac{2M}{2^n} \leq \frac{2M}{2^N} < \epsilon$$

for all $n \in \mathbb{N}$ and $\{x_{i_n}\} \to \alpha$. ∎

10.2.1 Bounding and the Extreme Value Theorem

Theorem 10.7 (Bounding Theorem) *If $f\colon [a, b] \to \mathbb{R}$ is continuous, then there is $M \in \mathbb{R}$ such that $|f(x)| \leq M$ for all $x \in [a, b]$.*

PROOF: Suppose this is not true. Then for any $n \in \mathbb{N}$ we can find $x_n \in [a,b]$ such that $|f(x_n)| > n$. By the Bolzano-Weierstrass Theorem, we can find a convergent subsequence $x_{i_1}, x_{i_2}, \ldots$. Note that $|f(x_{i_n})| > i_n \geq n$. Thus, replacing $\{x_n\}$ by $\{x_{i_n}\}$, we may, without loss of generality, assume that $\{x_n\}$ is convergent. So suppose $\{xn\} \to x$. Then $x \in [a,b]$ and $\{f(x_n)\} \to f(x)$. Now the sequence $\{f(x_n)\}$ is unbounded, and hence divergent, a contradication. ∎

Theorem 10.8 (Extreme Value Theorem) *Suppose $a < b$. If $f\colon [a,b] \to \mathbb{R}$, then there are $c, d \in [a,b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a,b]$.*

PROOF: Let $A = \{f(x) \mid a \leq x \leq b\}$. Then $A \neq \emptyset$ and, by the Bounding Theorem, A is bounded above and below. Let $\alpha = \sup A$. We claim that there is $d \in [a,b]$ with $f(d) = \alpha$.

Since $\alpha = \sup A$, for each $n \in \mathbb{N}$, there is $x_n \in [a,b]$ with $\alpha - \frac{1}{n} < f(x_n) \leq \alpha$. Note that $\{f(x_n)\}$ converges to α. By the Bolzano-Weierstrass Theorem, we can find a convergent subsequence. Replacing $\{x_n\}$ by a subsequence if necessary, we may assume $\{x_n\} \to d$ for some $d \in [a,b]$. Then $\{f(x_n)\} \to f(d)$. Thus $f(d) = \alpha$. Note that $f(x) \leq \alpha = f(d)$ for all $x \in [a,b]$.

Similarly, we can find $c \in [a,b]$ with $f(c) = \beta = \inf A$ and $f(c) \leq f(x)$ for all $x \in [a,b]$. ∎

10.2.2 Intermediate Value Theorem

Theorem 10.9 (Intermediate Value Theorem) *If $f\colon [a,b] \to \mathbb{R}$ is continuous and $f(a) < 0 < f(b)$, then there is $a < c < b$ with $f(c) = 0$.*

PROOF: We start to build a sequence of intervals

$$I_0 \supseteq I1 \supseteq I2 \supseteq \ldots$$

such that $I_n = [a_n, b_n]$, $f(a_n) < 0 < f(b_n)$ and $b_n - a_n = \frac{b-a}{2^n}$. Let $a_0 = a$, $b_0 = b$ and $I_0 = [a_0, b_0]$. Then $f(a_0) < 0 < f(b_0)$ and $b - a = (b-a)/2^0$.

Suppose we are have $I_n = [a_n, b_n]$ with $f(a_n) < 0 < f(b_n)$ and $b_n - a_n = (b-a)/2^n$. Let $d = (b_n - a_n)/2$. If $f(d) = 0$, then we have found $a < d < b$ with $f(d) = 0$ and are done. If $f(d) > 0$, let $a_{n+1} = a_n$ and $b_{n+1} = d$. If $f(d) < 0$, let $a_{n+1} = d$ and $b_{n+1} - b_n$.

Let $I_{n+1} = [a_{n+1}, b_{n+1}]$. Then $I_{n+1} \subset I_n$, $f(a_{n+1}) < 0 < f(b_{n+1})$ and $b_{n+1} - a_{n+1} = (b-a)/2^n$. By the Nested Interval Theorem, there is $c \in \bigcap_{n=0}^{\infty} I_n$. We claim that $f(c) = 0$.

Since $a_n, c \in I_n$, we know that $|a_n - c| \leq (b-a)/2^n$ for all $n \in \mathbb{N}$. If $\epsilon > 0$ is given we can find an N such that $(b-a)/2^N < \epsilon$. Then $|a_n - c| < \epsilon$ for all $n \in \mathbb{N}$. Hence $\{a_n\}$ converges to c. Thus, since f is continuous $\{f(a_n)\}$ converges to $f(c)$. Since $f(a_n) \leq 0$ for all n, we must have $f(c) \leq 0$.

Similarly, $\{b_n\} \to c$ and $\{f(b_n)\} \to f(c)$. But each $f(b_n) > 0$, thus $f(c) \geq 0$. Hence $f(c) = 0$. Thus there is $a < c < b$ with $f(c) = 0$. ∎

Corollary 10.1 *If $f\colon \mathbb{R} \to \mathbb{R}$ is a continuous function and $I \subset \mathbb{R}$ is an interval so that $I \subseteq \operatorname{dom}(f)$, then the set $f(I) = \{f(x) \mid x \in I\}$ is also an interval or a single point.*

PROOF: The set $J = f(I)$ has the property that if $y_0, y_1 \in J$ and $y_0 < y < y_1$, then $y \in J$ by the Intermediate Value Theorem.

If $\operatorname{glb} J < \operatorname{lub} J$ then J must be an interval. ∎