SERIES

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1. Determine if each of the following sequences converges or diverges. Find the limit if requested.

   (a) $\lim_{n\to\infty} \frac{n!}{n^n} = 0$

   (b) Verify that $\lim_{n\to\infty} \sqrt[n]{a^n + b^n} = \max(a, b)$
   Whichever is larger, a or b will dominate the sum $a^n + b^n$.

   (c) $\lim_{n\to\infty} n - \sqrt{n+a}\sqrt{n+b}$

$$\begin{aligned}\lim_{n\to\infty} n - \sqrt{n+a}\sqrt{n+b} &= \lim_{n\to\infty} n - \sqrt{n+a}\sqrt{n+b} \cdot \frac{n + \sqrt{n+a}\sqrt{n+b}}{n + \sqrt{n+a}\sqrt{n+b}} \\ &= \frac{-na - nb - ab}{n + \sqrt{n+a}\sqrt{n+b}} \\ &= \frac{-a - b - \frac{ab}{n}}{1 + \sqrt{1 + \frac{a}{n}}\sqrt{1 + \frac{b}{n}}} \\ &= -\left(\frac{a}{2} + \frac{b}{2}\right)\end{aligned}$$

   (d) $\lim_{n\to\infty} \frac{(-1)^n \sqrt{n} \sin(n^n)}{n+1}$

$$\lim_{n\to\infty} \frac{(-1)^n \sqrt{n} \sin(n^n)}{n+1} \leq \lim_{n\to\infty} \frac{\sqrt{n}}{n+1} = 0.$$

   (e) $\lim_{n\to\infty} \frac{a^n - b^n}{a^n + b^n}$

$$\lim_{n\to\infty} \frac{a^n - b^n}{a^n + b^n} = \begin{cases} 0 & \text{if } a = b \neq 0 \\ 1 & \text{if } |a| > |b| \\ -1 & \text{if } |a| < |b| \\ \text{undefined} & \text{if } a = b = 0 \end{cases}$$

   (f) $\lim_{n\to\infty} nc^n = 0$, $|c| < 1$

   (g) $\lim_{n\to\infty} \frac{2^{n^2}}{n!} = \infty$

   (h) $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2a_n}$. Find $\lim_{n\to\infty} a_n$
   Let $\ell = \lim a_n$, then $\lim \sqrt{2a_n} = \sqrt{2\ell}$. But $\lim \sqrt{2a_n} = \lim a_n$, so

$$\begin{aligned}\ell &= \sqrt{2\ell} \\ \ell^2 &= 2\ell \\ \ell(\ell - 2) &= 0\end{aligned}$$

   Thus, $\ell = 0$ or $\ell = 2$. Since a_n is increasing, the limit must be 2.

2. Decide whether each of the following infinite series is convergent or divergent. What test did you use?

   (a) $\sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$

   Since $\sin(n\theta) \leq 1, \frac{\sin n\theta}{n^2}, \frac{1}{n^2}$, so this series converges by the Comparison Test.

   (b) $\sum_{n=1}^{\infty} (-1)^n \frac{\log n}{n}$

   Since $\lim \frac{\log n}{n} = 0$, by the Alternating Series Test this series converges.

   (c) $\sum_{n=2}^{\infty} \frac{1}{\sqrt[3]{n^2 - 1}}$

   By the Comparison Test with $1/n^{2/3}$ this diverges.

   (d) $\sum_{n=1}^{\infty} \frac{n^2}{n!}$ By the Ratio Test

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{(n+1)^2}{(n+1)!} \frac{n!}{n^2} = \lim_{n\to\infty} \frac{n+1}{n^2} = 0$$

   the series converges. We can compute this sum if we recall that $\sum_{n=0}^{\infty} \frac{1}{n!} = e$.

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{n^2}{n!} &= \sum_{n=1}^{\infty} \frac{n}{(n-1)!} \\ &= \sum_{n=0}^{\infty} \frac{n+1}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=1}^{\infty} \frac{1}{(n-1)!} = \sum_{n=0}^{\infty} \frac{1}{n!} + \sum_{n=0}^{\infty} \frac{1}{n!} = 2e \end{aligned}$$

   (e) $\sum_{n=1}^{\infty} \frac{\log n}{n}$

   For $n > 3$, $\log n > 1$ so $\frac{\log n}{n} > \frac{1}{n}$ and this diverges by the Comparison Test.

   (f) $\sum_{n=2}^{\infty} \frac{1}{\log n}$

   $\log n < n$ for all n, so $\frac{1}{\log n} > \frac{1}{n}$ and this diverges by the Comparison Test.

   (g) $\sum_{n=2}^{\infty} \frac{1}{(\log n)^k}$, $k < n$

   The natural logarithm grows more slowly than any power of n. Therefore, eventually, $(\log n)^k < n$, which will force the series to diverge by the Comparison Test with $\sum \frac{1}{n}$.

(h) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(\log n)^n}$

Use the Root Test and $\displaystyle\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{1}{\log n} = 0$, so this series converges.

(i) $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{1}{(\log n)^n}$

This converges by the Alternating Series Test.

(j) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^3+1}$

Diverges by the Limit Comparison Test with $\sum \frac{1}{n}$.

(k) $\displaystyle\sum_{n=1}^{\infty} \sin\frac{1}{n}$

Use the Limit Comparison Test with $\sum \frac{1}{n}$.

$$\lim_{n\to\infty} \frac{\sin(\frac{1}{n})}{\frac{1}{n}} = \lim_{x\to 0} \frac{\sin x}{x} = 1 < \infty.$$

Thus, this series diverges.

(l) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2(\log n)}$

By the Comparison Test with $\sum \frac{1}{n^2}$ we have $n^2 \log n > n^2$, so $\frac{1}{n^2 \log n} < \frac{1}{n^2}$ and it converges.

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

Diverges, but grows slowly.