MATH 6101-090

ASSIGNMENT 3 - SOLUTIONS

25-September-2006

1. *Let (a,b) and (c,d) be any two open intervals in the real line.*

   (a) *Find a one-to-one function that maps $(0,1)$ to $(-1,1)$*

   There are many. We could take the linear function that maps $(0,1)$ to $(-1,1)$ that sends 0 to -1 and 1 to 1. That would be the line passing through $(0,-1)$ and $(1,1)$. It has slope 2 and y-intercept -1. Thus the equation is $f(x) = 2x - 1$.

   Likewise, you could take the cosine function which is one-to-one on the interval from $(0,\pi)$ and takes values in $(-1,1)$. To make it map $(0,1)$ to $(-1,1)$, simply take $f(x) = \cos(\pi x)$.

   (b) *Find a one-to-one function from (a,b) to (c,d).*

   The easiest one-to-one function would be a linear function that sends a to c and b to d. It is just a dilation and translation.

   The graph passes through the points (a,c) and (b,d) in the plane. Thus, the slope is $\dfrac{c-d}{a-b}$. The equation would then be

$$\begin{aligned} y - c &= \frac{c-d}{a-b}(x-a) \\ y &= \frac{c-d}{a-b}(x-a) + c \end{aligned}$$

   To show that it is one-to-one is to merely remark that it is a linear function and we have already shown that linear functions are one-one-one.

   (c) *Prove that any two open intervals in the real line have the same cardinality.*

   We just showed that there is a one-to-one function between any two open intervals. This is the necessary one-to-one correspondence between the two, so they have the same cardinality.

2. *Let $f(x) = 1/(1+x)$. What is*

   (a) $f(f(x))$ *(for which x does this make sense?)*

$$f(f(x)) = \frac{1}{1+f(x)} = \frac{1}{1+\frac{1}{1+x}} = \frac{1+x}{2+x}.$$

   The remaining question is does this make sense if $x = -1$? Clearly, we can substitute $x = -1$ into the final formula, but $f(f(-1))$ requires that we substitute -1 into $f(x)$ first, and that is undefined. Thus, the function is defined for all x, $x \neq -1, -2$.

   (b) $f(\frac{1}{x})$?

$$f\left(\frac{1}{x}\right) = \frac{1}{1+\frac{1}{x}} = \frac{x}{1+x}.$$

(c) $f(cx)$?

$$f(cx) = \frac{1}{1+cx}.$$

(d) $f(x+y)$?

$$f(x+y) = \frac{1}{1+x+y}.$$

(e) $f(x)+f(y)$?

$$f(x)+f(y) = \frac{1}{1+x} + \frac{1}{1+y} = \frac{2+x+y}{(1+x)(1+y)}.$$

(f) *For which numbers c is there a number x so that $f(cx) = f(x)$?*
For $f(cx) = f(x)$ we would have

$$f(cx) = \frac{1}{1+cx} = \frac{1}{1+x} = f(x).$$

We only need to have one x where this is true and for $x = 0$ it does not matter what value c takes (as long as it is a real number), we would have that $f(c \cdot 0) = 1 = f(0)$. Thus, for every real number c there is a number $x = 0$ such that $f(cx) = f(x)$.

(g) *For which numbers c is it true that $f(cx) = f(x)$ for two different numbers x?*
Clearly, if $c = 1$ this is true for all $x \neq -1$. Let $x \neq 0$ and let $f(cx) = f(x)$. We have

$$\begin{aligned} f(cx) &= f(x) \\ \frac{1}{1+cx} &= \frac{1}{1+x} \\ 1+cx &= 1+x \\ x(c-1) &= 0 \end{aligned}$$

If $x \neq 0$, then $c = 1$. Thus for $f(cx) = f(x)$ for two different numbers x we must have $c = 1$.

3. *For which numbers a, b, c, d will the function*

$$f(x) = ax + bcx + d$$

satisfy $f(f(x)) = x$ for all x?

Checking the algebra:

$$\begin{aligned} f(f(x)) &= \frac{af(x)+b}{cf(x)+d} \\ &= \frac{a\frac{ax+b}{cx+d}+b}{c\frac{ax+b}{cx+d}+d} \\ &= \frac{(a^2+bc)x+(ab+bd)}{(ac+cd)x+(bc+d^2)} \end{aligned}$$

Setting $f(f(x)) = x$ gives

$$\begin{aligned} x &= \frac{(a^2+bc)x + (ab+bd)}{(ac+cd)x + (bc+d^2)} \\ x\left((ac+cd)x + (bc+d^2)\right) &= (a^2+bc)x + (ab+bd) \\ (ac+cd)x^2 + (bc+d^2)x &= (a^2+bc)x + (ab+bd) \end{aligned}$$

This gives us that

$$\begin{aligned} ac+cd &= 0 \\ bc+d^2 &= a^2+bc \\ ab+bd &= 0 \end{aligned}$$

All of these give us that $a = -d$, $b \neq 0$, and $c \neq 0$.

4. *Suppose that H is a function.*

   (a) *Suppose that y is a number such that $H(H(y)) = y$, what is*

   $$\underbrace{H(H(H(\cdots(H(y)\cdots))))}_{20 \text{ times}}?$$

   Since $H(H(y)) = y$, every pair of H's gives us the identity map. There are 10 pairs in the above composition, so

   $$\underbrace{H(H(H(\cdots(H(y)\cdots))))}_{20 \text{ times}} = y.$$

   (b) *Same question if 20 is replaced by 21.*
   This time there are an odd number of pairs, so

   $$\underbrace{H(H(H(\cdots(H(y)\cdots))))}_{21 \text{ times}} = H(y).$$

   (c) *Same question if $H(H(y)) = H(y)$.* This time every pair of H's gives us H back. Thus in both cases

   $$\underbrace{H(H(H(\cdots(H(y)\cdots))))}_{20 \text{ times}} = \underbrace{H(H(H(\cdots(H(y)\cdots))))}_{21 \text{ times}} = H(y).$$

5. *Let f and g be functions $f, g\colon \mathbb{R} \to \mathbb{R}$.*

   (a) *Determine whether $f+g$ is even, odd, or neither in the four cases obtained by choosing f even or odd and g even or odd.*

   (b) *Do the same for $f \cdot g$.*

   (c) *Do the same for $f \circ g$.*

We will do them all at the same time.

$f+g$	f even	f odd
g even	even	neither
g odd	neither	odd

$f \cdot g$	f even	f odd
g even	even	odd
g odd	odd	even

$f \circ g$	f even	f odd
g even	even	even
g odd	even	odd

Case I f and g are both even, *i.e.*, $f(-x) = f(x)$ and $g(-x) = g(x)$. Then,

$$\begin{aligned}
(f+g)(-x) &= f(-x)+g(-x) = f(x)+g(x) = (f+g)(x) \text{ is even.}\\
(f \cdot g)(-x) &= f(-x) \cdot g(-x) = f(x) \cdot g(x) = (f \cdot g)(x) \text{ is even.}\\
(f \circ g)(-x) &= f(g(-x)) = f(g(x)) = (f \circ g)(x) \text{ is even.}
\end{aligned}$$

Case II f is even and g is odd, *i.e.*, $f(-x) = f(x)$ and $g(-x) = -g(x)$. Then,

$$\begin{aligned}
(f+g)(-x) &= f(-x)+g(-x) = f(x)-g(x) \text{ is neither.}\\
(f \cdot g)(-x) &= f(-x) \cdot g(-x) = f(x) \cdot (-g(x)) = -f(x) \cdot g(x) = -(f \cdot g)(x) \text{ is odd.}\\
(f \circ g)(-x) &= f(g(-x)) = f(-g(x)) = f(g(x)) = (f \circ g)(x) \text{ is even.}
\end{aligned}$$

Case III f is odd and g is even, *i.e.*, $f(-x) = -f(x)$ and $g(-x) = g(x)$. Then,

$$\begin{aligned}
(f+g)(-x) &= f(-x)+g(-x) = -f(x)+g(x) \text{ is neither.}\\
(f \cdot g)(-x) &= f(-x) \cdot g(-x) = -f(x) \cdot g(x) = -(f \cdot g)(x) \text{ is odd.}\\
(f \circ g)(-x) &= f(g(-x)) = f(g(x)) = (f \circ g)(x) \text{ is even.}
\end{aligned}$$

Case IV f and g are both odd, *i.e.*, $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Then,

$$\begin{aligned}
(f+g)(-x) &= f(-x)+g(-x) = -f(x)-g(x) = -(f+g)(x) \text{ is odd.}\\
(f \cdot g)(-x) &= f(-x) \cdot g(-x) = -f(x) \cdot (-g(x)) = (f \cdot g)(x) \text{ is even.}\\
(f \circ g)(-x) &= f(g(-x)) = f(-g(x)) = -f(g(x)) = (f \circ g)(x) \text{ is odd.}
\end{aligned}$$

6. *Let f, g, and h be functions from the reals to the reals. Prove or give a counterexample to each of the following.*

   (a) $f \circ (g+h) = f \circ g + f \circ h$

   False, let $f(x) = x^2$ and g and h be arbitrary, but neither is the constant function 0. Then,

$$\begin{aligned}
f \circ (g+h)(x) &= f((g+h)(x)) = f(g(x)+h(x))\\
&= (g(x)+h(x))^2 = (g(x))^2 + 2g(x)h(x) + (h(x))^2\\
f \circ (g+h)(x) &\neq (g(x))^2 + (h(x))^2 = (f \circ g)(x) + (f \circ h)(x).
\end{aligned}$$

   (b) $(g+h) \circ f = g \circ f + h \circ f$

   True,

$$((g+h) \circ f)(x) = (g+h)(f(x)) = g(f(x)) + h(f(x)) = (g \circ f)(x) + (h \circ f)(x).$$

(c) $\dfrac{1}{f \circ g} = \dfrac{1}{f} \circ g$
True,

$$\left(\frac{1}{f \circ g}\right)(x) = \frac{1}{f(g(x))} = \left(\frac{1}{f}\right)(g(x)) = \left(\frac{1}{f} \circ g\right)(x).$$

(d) $\dfrac{1}{f \circ g} = f \circ \dfrac{1}{g}$
This one is false. Let $f(x) = x^2 + 1$ and let $g(x)$ be any non-zero function.

$$\left(\frac{1}{f \circ g}\right)(x) = \frac{1}{f(g(x))} = \frac{1}{g(x)^2 + 1},$$

while

$$\left(f \circ \frac{1}{g}\right)(x) = f\left(\frac{1}{g(x)}\right) = \frac{1}{g(x)^2} + 1.$$

These two functions are different for all choices of g.