MATH 6101-090

ASSIGNMENT 4 - SOLUTIONS

02-October-2006

1. *Find the limits of the following sequences*

   (a) $a_n = \sqrt{n^2+1} - n$.

$$a_n = \sqrt{n^2+1} - n = \frac{1}{\sqrt{n^2+1}+n}\frac{\sqrt{n^2+1}-n}{\sqrt{n^2+1}-n} = \frac{1}{\sqrt{n^2+1}+n}$$

To find this limit we will multiply both top and bottom by $1/n$.

$$\frac{1}{\sqrt{n^2+1}+n} = \frac{\frac{1}{n}}{\frac{1}{n}\sqrt{n^2+1}+1} = \frac{\frac{1}{n}}{\sqrt{1+\frac{1}{n^2}}+1}$$

As $n \to \infty$ the top goes to 0 and the bottom goes to $1+1=2$, so the term approaches 0.

$$\lim_{n\to\infty} a_n = 0.$$

   (b) $b_n = \sqrt{n^2+n} - n$.

We will do something similar here. We need to modify the fraction — in a sense we need to "irrationalize" the denominator.

$$\sqrt{n^2+n} - n = \frac{n}{\sqrt{n^2+n}+n}\frac{\sqrt{n^2+n}-n}{\sqrt{n^2+n}-n} = \frac{n}{\sqrt{n^2+n}+n}$$

So,

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}+n} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{1}{n}\sqrt{n^2+n}+1} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+1} = \frac{1}{2}
\end{aligned}$$

   (c) $c_n = \sqrt{4n^2+n} - 2n$. Again, irrationalizing the denominator, we have

$$\sqrt{4n^2+n} - 2n = \frac{n}{\sqrt{4n^2+n}+2n}\frac{\sqrt{4n^2+n}-2n}{\sqrt{4n^2+n}-2n},$$

so,

$$\begin{aligned}
\lim_{n\to\infty} c_n &= \lim_{n\to\infty} \frac{n}{\sqrt{4n^2+n}+2n} \\
&= \lim_{n\to\infty} \frac{\frac{n}{n}}{\frac{1}{n}\sqrt{4n^2+n}+2} \\
&= \lim_{n\to\infty} \frac{1}{\sqrt{4+\frac{1}{n}}+2} = \frac{1}{4}
\end{aligned}$$

2. *Suppose that* $\lim x_n = 3$, $\lim y_n = 7$ *and that all* y_n *are nonzero. Determine the following limits:*

$$(a)\lim(x_n + y_n) \qquad (b)\lim\frac{3y_n - x_n}{y_n^2}$$

(a) $\lim(x_n + y_n) = \lim x_n + \lim y_n = 3 + 7 = 10.$

(b)

$$\lim\frac{3y_n - x_n}{y_n^2} = \frac{3\lim y_n - \lim x_n}{(\lim y_n)^2} = \frac{3\cdot 7 - 3}{7^2} = \frac{18}{49}.$$

3. *Let* $a_1 = 1$ *and for* $n \geq 1$ *let* $a_{n+1} = \sqrt{a_n + 1}$.

(a) *List the first five terms of* $\{a_n\}$.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \sqrt{2} \\ a_3 &= \sqrt{1+\sqrt{2}} \\ a_4 &= \sqrt{1+\sqrt{1+\sqrt{2}}} \\ a_5 &= \sqrt{1+\sqrt{1+\sqrt{1+\sqrt{2}}}} \end{aligned}$$

(b) *It turns out that* $\{a_n\}$ *converges. Assume that this is true and show that the limit is* $\frac{1}{2}(1+\sqrt{5})$.

Since $\{a_n\}$ converges, its limit a must satisfy the equation $a = \sqrt{a+1}$. What this means is that

$$\begin{aligned} \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \sqrt{a_n + 1} \\ \lim_{n\to\infty} a_{n+1} &= \sqrt{\lim_{n\to\infty} a_n + 1} \\ a &= \sqrt{a+1} \end{aligned}$$

This is only true because the limit exists!. This means

$$\begin{aligned} a &= \sqrt{a+1} \\ a^2 &= a+1 \\ a^2 - a - 1 &= 0 \end{aligned}$$

The roots to this equation are $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$. Now, all of the a_n's are positive, so the sequence must converge to the positive root, or

$$\lim_{n\to\infty} a_n = \frac{1+\sqrt{5}}{2}.$$

4. *Let* $a_1 = 1$ *and* $a_{n+1} = \frac{1}{3}(a_n + 1)$ *for* $n \geq 1$.

   (a) Find a_2, a_3, a_4 and a_5.

$$\begin{aligned} a_1 &= 1 \\ a_2 &= \frac{1}{3}(1+1) = \frac{2}{3} \\ a_3 &= \frac{1}{3}\left(\frac{2}{3}+1\right) = \frac{5}{9} \\ a_4 &= \frac{1}{3}\left(\frac{5}{9}+1\right) = \frac{14}{27} \\ a_5 &= \frac{1}{3}\left(\frac{14}{27}+1\right) = \frac{41}{81} \end{aligned}$$

   (b) *Use induction to show that* $a_n > \frac{1}{2}$ *for all* n.

Method I:

First, $a_1 = 1 > 1/2$ so the statement is true for n=1. Assume that the statement is true for n, *i*.e. assume that $a_n > \frac{1}{2}$. We need to show that $a_{n+1} > \frac{1}{2}$.

$$\begin{aligned} a_{n+1} &= \frac{1}{3}(a_n + 1) \\ a_{n+1} &> \frac{1}{3}\left(\frac{1}{2}+1\right) = \frac{1}{3}\cdot\frac{3}{2} = \frac{1}{2} \end{aligned}$$

Thus, we have shown that if it is true for n it is also true for $n+1$, and the statement is true by induction.

Method II:

Checking you have that

$$\begin{aligned} a_n &= \frac{2+3+9+27+\cdots+3^{n-2}}{3^{n-1}} \\ &= \frac{2+\sum_{k=1}^{n-2} 3^k}{3^{n-1}} \\ &= \frac{2+\frac{3^{n-1}-1}{3-1}-1}{3^{n-1}} \\ &= \frac{1}{2}\left(1+\frac{1}{3^{n-1}}\right) \end{aligned}$$

We need to show this is true for all n.

$$\begin{aligned} a_{n+1} &= \frac{1}{3}(a_n+1) = \frac{1}{3}\left(1+\frac{1}{2}\left(1+\frac{1}{3^{n-1}}\right)\right) \\ &= \frac{1}{3}\frac{1}{2}\left(3+\frac{1}{3^{n-1}}\right) \\ &= \frac{1}{2}\left(1+\frac{1}{3^n}\right) \end{aligned}$$

which is what we needed to show. Now,

$$a_n = \frac{1}{2} + \frac{1}{2}\frac{1}{3^{n-1}} > \frac{1}{2}.$$

(c) *Show that* $\{a_n\}$ *is a nonincreasing sequence.*
Method I:
We need to show that $a_n > a_{n+1}$ for all n. We will be done if we can show that $a_n - a_{n+1} > 0$ for all n, since the first statement clearly follows from the second.

$$\begin{aligned} a_n - a_{n+1} &= a_n - \frac{1}{3}(a_n + 1) \\ a_n - a_{n+1} &= \frac{2}{3}a_n - \frac{1}{3} > \frac{2}{3}\cdot\frac{1}{2} - \frac{1}{3} = 0 \end{aligned}$$

and we have shown what we needed to show.
Method II:

$$a_{n+1} = \frac{1}{2}\left(1 + \frac{1}{3^n}\right) < \frac{1}{2}\left(1 + \frac{1}{3^{n-1}}\right) = a_n,$$

so each term is smaller than the previous and the sequence is nonincreasing.

(d) *Find* $\lim a_n$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\frac{1}{2}\left(1 + \frac{1}{3^{n-1}}\right) = \frac{1}{2}.$$

5. *For each of the following sequences find the* $\operatorname{glb}\{a_n\}$, $\operatorname{lub}\{a_n\}$, $\limsup\{a_n\}$, *and* $\liminf\{a_n\}$.

(a) $\{(-1)^n\}_{n=0}^{\infty}$ Let A denote the set of values of this sequence. $A = \{-1, 1\}$. Thus, $\operatorname{lub} A = 1$, $\operatorname{glb} A = -1$, $\liminf\{a_n\} = -1$, and $\limsup\{a_n\} = 1$.

(b) $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ Let B denote the set of values of this sequence.

$$B = \left\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\right\}.$$

From this we see that $\operatorname{lub} B = \operatorname{lub}\{a_n\} = 1$ and $\operatorname{glb} B = \operatorname{glb}\{a_n\} = 0$. The "tails" of this sequence are $A_n = \{\frac{1}{n}, \frac{1}{n+1}, \ldots\}$. Then $u_n = \operatorname{lub} A_n = \frac{1}{n}$ and $v_n = \operatorname{glb} A_n = 0$. Then the lim sup and lim inf are

$$\begin{aligned} \limsup\{a_n\} &= \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{n} = 0 \\ \liminf\{a_n\} &= \lim_{n\to\infty} v_n = \lim_{n\to\infty} 0 = 0 \end{aligned}$$

(c) $\{(-1)^n n\}_{n=0}^{\infty}$ Let B denote the set of values of this sequence.

$$B = \{0, -1, 2, -3, 4, -5, 6, \ldots\}.$$

From this we see that $\operatorname{lub} B = \operatorname{lub}\{a_n\} = +\infty$ and $\operatorname{glb} B = \operatorname{glb}\{a_n\} = -\infty$. The "tails" of this sequence are $A_n = \{(-1)^n n, (-1)^{n+1}(n+1), \ldots\}$. Then $u_n = \operatorname{lub} A_n = +\infty$ and $v_n = \operatorname{glb} A_n = -\infty$. Then the lim sup and lim inf are

$$\begin{aligned} \limsup\{a_n\} &= \lim_{n\to\infty} u_n = +\infty \\ \liminf\{a_n\} &= \lim_{n\to\infty} v_n = -\infty \end{aligned}$$

6. *Let $\{a_n\}$ and $\{b_n\}$ be the following sequences that repeat in cycles of four.*

$$\begin{aligned} \{a_n\} &= \{0,1,2,1,0,1,2,1,0,1,2,1,0,1,2,1,0,\ldots\} \\ \{b_n\} &= \{2,1,1,0,2,1,1,0,2,1,1,0,2,1,1,0,2,\ldots\} \end{aligned}$$

First, let's find a few items:

$$\begin{aligned} \{a_n + b_n\} &= \{2,2,3,1,2,2,3,1,2,2,3,1,2,2,3,1,\ldots\} \\ \{a_n b_n\} &= \{0,1,2,0,0,1,2,0,0,1,2,0,0,1,2,0,\ldots\} \\ \liminf a_n &= 0 \\ \limsup a_n &= 2 \\ \liminf b_n &= 0 \\ \limsup b_n &= 2 \end{aligned}$$

(a) $\liminf a_n + \liminf b_n = 0 + 0 = 0$

(b) $\liminf(a_n + b_n) = 1$

(c) $\liminf a_n + \limsup b_n = 0 + 2 = 2$

(d) $\limsup(a_n + b_n) = 3$

(e) $\limsup a_n + \limsup b_n = 2 + 2 = 4$

(f) $\liminf a_n b_n = 0$

(g) $\limsup a_n b_n = 2$