# MATH 6101-090

## *ASSIGNMENT 5 - SOLUTIONS*

23-October-2006

1. *Use the limit convergence test to decide whether the following series converge or diverge. Note that you need to know convergence of the p-series.*

   (a) *Does the series* $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$ *converge or diverge?*

   Using the Limit Comparison Test we can see that by comparing the given series to the series $\sum 1/n^2$:

$$r = \lim_{n\to\infty} \frac{\dfrac{n+5}{n^3-2n+3}}{\dfrac{1}{n^2}} = \lim_{n\to\infty} \frac{n^3+5n^2}{n^3-2n+3} = 1.$$

   Since $0 < r < +\infty$ and since $\sum 1/n^2$ converges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{n+5}{n^3-2n+3}$.

   (b) *Does the series* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ *converge of diverge?*

   This time use the Limit Comparison Test with the series $\sum 1/\sqrt{n}$.

$$r = \lim_{n\to\infty} \frac{\dfrac{1}{1+\sqrt{n}}}{\dfrac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1.$$

   Since $0 < r < +\infty$ and since $\sum 1/\sqrt{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$.

2. (a) *What is the actual limit of the sum* $\displaystyle\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n$*?*

   Use the geometric series to find this sum.

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \left(\frac{1}{2}\right)^2 \left(\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n\right) = \frac{1}{4} \cdot \frac{1}{1-\frac{1}{2}} = \frac{1}{2}.$$

   Alternatively, you could do this by:

$$\sum_{n=2}^{\infty} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n - \left[\left(\frac{1}{2}\right)^0 + \left(\frac{1}{2}\right)^1\right] = \frac{1}{1-\frac{1}{2}} - 1 - \frac{1}{2} = \frac{1}{2}.$$

   (b) *What is the actual limit of the sum* $\displaystyle\sum_{n=10}^{\infty} \left(\frac{3}{4}\right)^n$*?*

Do this one like above:

$$\sum_{n=10}^{\infty}\left(\frac{3}{4}\right)^n = \left(\frac{3}{4}\right)^{10}\left(\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n\right) = \frac{59049}{1048576}\cdot\frac{1}{1-\frac{3}{4}} = \frac{59049}{262144}.$$

The alternative way does not look as appealing this time:

$$\sum_{n=10}^{\infty}\left(\frac{3}{4}\right)^n = \sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n - \sum_{n=0}^{9}\left(\frac{3}{4}\right)^n = \frac{1}{1-\frac{3}{4}} - \frac{1-(\frac{3}{4})^{10}}{1-\frac{3}{4}} = 4 - \frac{989527}{262144} = \frac{59049}{262144}.$$

(c) *Does the sum* $\displaystyle\sum_{n=1}^{\infty}\frac{2^n+1}{3^n-4}$ *converge? What test do you use to determine convergence or divergence?*

This series converges using the Limit Comparison Test with the geometric series $\sum(\frac{2}{3})^n$.

3. *Show that if* $\sum a_n$ *and* $\sum b_n$ *are convergent series of nonnegative numbers, then* $\sum\sqrt{a_nb_n}$ *converges.* [HINT: Show that $\sqrt{a_nb_n} \le a_n + b_n$.]

First, to show the hint,

$$\begin{aligned}(\sqrt{a_n}-\sqrt{b_n})^2 &\ge 0\\ a_n - 2\sqrt{a_n}\sqrt{b_n} + b_n &\ge 0\\ a_n + b_n &\ge 2\sqrt{a_nb_n} \ge \sqrt{a_nb_n}\end{aligned}$$

Now since both $\sum a_n$ and $\sum b_n$ are convergent series, then their sum, $\sum(a_n+b_n)$ is a convergent series. We have just shown that $\sqrt{a_nb_n} \le a_n + b_n$, so by the Comparison Test $\sum\sqrt{a_nb_n}$ converges.

4. *Determine which of the following series converge and justify your answer.*

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n^4}{2^n}$ We should probably try the Ratio Test:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)^4}{2^{n+1}}\cdot\frac{2^n}{n^4} = \lim_{n\to\infty}\frac{1}{2}\cdot\left(\frac{n+1}{n}\right)^4 = \frac{1}{2}.$$

Since the ratio is less than 1, the series converges. In fact we can show that $\displaystyle\sum_{n=1}^{\infty}\frac{n^4}{2^n} = 150.$ How about that?

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n!}$ Let's try the Ratio Test again.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^{n+1}}{(n+1)!}\cdot\frac{n!}{2^n} = \lim_{n\to\infty}\frac{2}{n+1} = 0.$$

Since the ratio is less than 1, the series converges. In fact we can show that $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n!} = e^2.$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^2}$ We will use the Comparison Test here. $\cos^2 n \leq 1$ for all $n$, so

$$\frac{\cos^2 n}{n^2} \leq \frac{1}{n^2}$$

for all $n$. Since $\sum 1/n^2$ converges, so does this series. We may not be able to evaluate this sum, but we know that it is less than $\pi^2/6$.

(d) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^n}$ This one is set up to use the Root Test.

$$\alpha = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left|\frac{1}{n^n}\right|^{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{n} = 0.$$

Since $\alpha < 1$ the series converges.

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{100^n}{n!}$ Let's try the Ratio Test again.

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \lim_{n\to\infty} \frac{100}{n+1} = 0.$$

Since the ratio is less than 1, the series converges. In fact we can show that $\displaystyle\sum_{n=1}^{\infty} \frac{100^n}{n!} = e^{100}$.

5. *We have seen that it is often harder to find the value of an infinite sum than to show that it exists. Here are some sums that you can find.*

(a) *Calculate* $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ *and* $\displaystyle\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n$.

These are geometric series, but they start at 1 not at 0. Thus the sum is $\dfrac{1}{1-r} - 1$.

$$\begin{aligned}
\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n &= \frac{1}{1-\frac{2}{3}} - 1 = 2 \\
\sum_{n=1}^{\infty} \left(-\frac{2}{3}\right)^n &= \frac{1}{1+\frac{2}{3}} - 1 = \frac{2}{3}
\end{aligned}$$

(b) *Prove that* $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$. *Compare assignment 2.*

In Assignment 2 we showed that $s_n = 1 - \frac{1}{n+1}$. Thus, the sum, which is the limit of the partial sums, is 1.

(c) *Prove that* $\displaystyle\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} = \frac{1}{2}$. [*Hint*: Note that $\dfrac{k-1}{2^{k+1}} = \dfrac{k}{2^k} - \dfrac{k+1}{2^{k+1}}$].

First,
$$\frac{k}{2^k} - \frac{k+1}{2^{k+1}} = \frac{2k-k-1}{2^{k+1}} = \frac{k-1}{2^{k+1}}.$$
Thus,
$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} &= \sum_{n=1}^{\infty} \left( \frac{n}{2^n} - \frac{n+1}{2^{n+1}} \right) \\
&= \sum_{n=1}^{\infty} \frac{n}{2^n} - \sum_{n=1}^{\infty} \frac{n+1}{2^{n+1}} \\
&= \frac{1}{2} + \sum_{n=2}^{\infty} \frac{n}{2^n} - \sum_{n=2}^{\infty} \frac{n}{2^n} = \frac{1}{2}
\end{aligned}$$

(d) *Use (c) to compute* $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$.

$$\begin{aligned}
\frac{1}{2} = \sum_{n=1}^{\infty} \frac{n-1}{2^{n+1}} &= \frac{1}{4} \sum_{n=1}^{\infty} \frac{n-1}{2^{n-1}} \\
&= \frac{1}{4} \sum_{n=0}^{\infty} \frac{n}{2^n} \\
\sum_{n=0}^{\infty} \frac{n}{2^n} &= \sum_{n=1}^{\infty} \frac{n}{2^n} = 4 \cdot \frac{1}{2} = 2
\end{aligned}$$