ASSIGNMENT 6
SOLUTIONS

30-October-2006

1. *Prove that* $n^2 < 2^n$ *for all* $n \geq 5$.

   First, let's check it for $n = 5$. $5^2 = 25 < 32 = 2^5$, so it is true for $n = 5$.

   Let us assume that $n^2 < 2^n$ and we need to show that this is true for $n + 1$.

   $$(n+1)^2 = n^2 + 2n + 1 < 2^n + 2n + 1.$$

   In order to complete the proof, we need to know that $2n + 1 < 2^n$ for $n > 5$. This is clearly true for $n = 5$, so assuming that it is true for n, let's check it for $n + 1$.

   $$2(n+1) + 1 = 2n + 1 + 2 < 2^n + 2 < 2^n + 2^n = 2^{n+1},$$

   so this is true for all $n > 5$. Thus, in our original proof, we now have:

   $$\begin{aligned}(n+1)^2 &= n^2 + 2n + 1 \\ &< 2^n + 2n + 1 \\ &< 2^n + 2^n = 2^{n+1};\end{aligned}$$

   and we are done.

2. *Consider the sequence*

   $$0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, \ldots$$

   where each string of zeros has one more zero than the previous. Does this sequence converge or diverge? If it converges to a limit L*, prove that it converges to* L*. If it diverges, prove that it diverges.*

   The sequence diverges. One way of showing this is that there is a subsequence which does not converge to 0. Another way of showing this is that regardless of what ϵ is given, for any $N > 0$ there is another $a_n = 1$ for $n > N$. This will keep the sequence from converging.

3. *Suppose* $f\colon \mathbb{R} \to \mathbb{R}$ *and* $f(x + y) = f(x) + f(y)$ *for all* $x, y \in \mathbb{R}$.

   (a) *Prove that* $f(0) = 0$.
   Since $f(0) = f(0 + 0) = f(0) + f(0)$ we get $f(0) = 0$.

   (b) *Prove by induction that* $f(nx) = nf(x)$ *for all* $x \in \mathbb{R}$ *and* $n \in \mathbb{N}$.
   First, we know that it is true for $n = 1$ because $f(1 \cdot x) = f(x) = 1 \cdot f(x)$. Assume this to be true for n. We need to show that $f((n + 1)x) = (n + 1)f(x)$.

   $$\begin{aligned}f((n+1)x) &= f(nx + x) \\ &= f(nx) + f(x) \\ &= nf(x) + f(x) = (n+1)f(x)\end{aligned}$$

   as we wanted.
   Let $\alpha = f(1)$.

   (c) *Prove that* $f(x) = \alpha x$ *for all* $x \in \mathbb{N}$.
   Let $x \in \mathbb{N}$, then $f(x) = f(x \cdot 1) = x \cdot f(1)$ by our previous part. Thus, $f(x) = \alpha x$ if $x \in \mathbb{N}$.

   (d) *Prove that* $f(-x) = -f(x)$ *for all* $x \in \mathbb{R}$*. Conclude that* $f(x) = \alpha x$ *for all* $x \in \mathbb{Z}$.
   To see that $f(-x) = -f(x)$, recall that $0 = f(0) = f(x - x) = f(x) + f(-x)$ which leads us to $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Since $f(x) = \alpha x$ if $x \in \mathbb{N}$, then if $x \in \mathbb{Z}$ and $x < 0$, then $x = -k$, where $k \in \mathbb{N}$. Thus, $f(x) = f(-k) = -f(k) = -k\alpha = x\alpha = \alpha x$ if $x \in \mathbb{Z}$.

(e) *Prove that* $f(\frac{x}{n}) = \frac{f(x)}{n}$ *for all* $x \in \mathbb{R}$. *Conclude that* $f(x) = \alpha x$ *for all* $x \in \mathbb{Q}$.
If $x \in \mathbb{R}$, then

$$f(x) = f\left(n \cdot \frac{x}{n}\right) = nf\left(\frac{x}{n}\right).$$

so $f(\frac{x}{n}) = \frac{f(x)}{n}$ for all $x \in \mathbb{R}$. If $x \in \mathbb{Q}$, then $x = \frac{p}{q}$ where $p, q \in \mathbb{Z}$. We know that

$$f(\frac{p}{q}) = \frac{f(p)}{q} = \frac{\alpha p}{q} = \alpha\frac{p}{q} = \alpha x.$$

(f) *Suppose in addition that* f *is continuous, i.e. that for all* $a \in \mathbb{R}$, $\lim_{x\to a} f(x) = f(a)$. *Prove that* $f(x) = \alpha x$ *for all* $x \in \mathbb{R}$.
Let $\{a_n\}$ be a sequence of rational numbers whose limit is x. Since the rationals are dense in $\mathbb{R}$ we can choose such a sequence. Then

$$f(x) = f(\lim a_n) = \lim f(a_n) = \lim \alpha a_n = \alpha \lim a_n = \alpha x.$$