ASSIGNMENT 8
SOLUTIONS

13-November-2006

1. *Express all the hyperbolic functions in terms of* $\sinh x$*. Given* $\cosh x = 2$ *find the values of the other functions.*

$$\begin{aligned}
\sinh x &= \sinh x \\
\cosh x &= \sqrt{\sinh^2 x + 1} \\
\tanh x &= \frac{\sinh x}{\sqrt{\sinh^2 x + 1}} \\
\coth x &= \frac{\sqrt{\sinh^2 x + 1}}{\sinh x} \\
\operatorname{sech} x &= \frac{1}{\sqrt{\sinh^2 x + 1}} \\
\operatorname{csch} x &= \frac{1}{\sinh x}
\end{aligned}$$

Now, if $\cosh x = 2$, then $\sinh x = \sqrt{3}$ and

$$\begin{aligned}
\sinh x &= \sqrt{3} \\
\cosh x &= 2 \\
\tanh x &= \frac{\sqrt{3}}{2} \\
\coth x &= \frac{2}{\sqrt{3}} \\
\operatorname{sech} x &= \frac{1}{2} \\
\operatorname{csch} x &= \frac{1}{\sqrt{3}}
\end{aligned}$$

2. (a) *Show that*

$$(\cosh u_1 + \sinh u_1)(\cosh u_2 + \sinh u_2) = \cosh(u_1 + u_2) + \sinh(u_1 + u_2).$$

$$\begin{aligned}
(\cosh u_1 + \sinh u_1)(\cosh u_2 + \sinh u_2) &= \cosh u_1 \cosh u_2 + \cosh u_1 \sinh u_2 + \\
&\quad \sinh u_1 \cosh u_2 + \sinh u_1 \sinh u_2 \\
&= (\cosh u_1 \cosh u_2 + \sinh u_1 \sinh u_2) + \\
&\quad (\sinh u_1 \cosh u_2 + \cosh u_1 \sinh u_2) \\
&= \cosh(u_1 + u_2) + \sinh(u_1 + u_2)
\end{aligned}$$

(b) *Show that for any positive integer $n > 0$*

$$\prod_{i=1}^{n}(\cosh u_i + \sinh u_i) = \cosh\left(\sum_{i=1}^{n} u_i\right) + \sinh\left(\sum_{i=1}^{n} u_i\right).$$

This is done by induction with (a) serving as the first step. Assume that this is true for n. We need to prove it true for $n+1$.

$$\begin{aligned}\prod_{i=1}^{n+1}(\cosh u_i + \sinh u_i) &= \left(\prod_{i=1}^{n}(\cosh u_i + \sinh u_i)\right)(\cosh u_{n+1} + \sinh u_{n+1})\\ &= \left(\cosh\left(\sum_{i=1}^{n} u_i\right) + \sinh\left(\sum_{i=1}^{n} u_i\right)\right)(\cosh u_{n+1} + \sinh u_{n+1})\end{aligned}$$

which by the above process

$$\begin{aligned}&= \cosh\left(\sum_{i=1}^{n} u_i + u_{n+1}\right) + \sinh\left(\sum_{i=1}^{n} u_i + u_{n+1}\right)\\ &= \cosh\left(\sum_{i=1}^{n+1} u_i\right) + \sinh\left(\sum_{i=1}^{n+1} u_i\right)\end{aligned}$$

which is what we needed to show.

(c) *What does this become if $u_1 = u_2 = \cdots = u_n = u$?*
This becomes an analogue of Euler's Formula:

$$(\cosh u + \sinh u)^n = \cosh(nu) + \sinh(nu).$$

3. *Evaluate the following integral in terms of hyperbolic trigonometric functions*

$$\int \frac{1}{\sqrt{4+x^2}}\,dx$$

Use the substitution $x = 2\sinh u$, then $dx = 2\cosh u\,du$ and the integral becomes:

$$\begin{aligned}\int \frac{1}{\sqrt{4+x^2}}\,dx &= \int \frac{1}{\sqrt{4\cosh^2 u}} 2\cosh u\,du\\ &= \int 1\,du = u = \sinh^{-1}\left(\frac{x}{2}\right) + C.\end{aligned}$$

4. *Differentiate the following functions.*

(a) $f(x) = 3x\tanh(4x)$.

$$f'(x) = 3\tanh(4x) + 12x\,\overset{2}{\text{sech}}(4x).$$

(b) $g(x) = 5x \operatorname{sech}(4x) - 21 \tanh^3(4x)$.

$$\begin{aligned} g'(x) &= 5 \operatorname{sech}(4x) - 20x \operatorname{sech}(4x) \tanh(4x) - 63 \tanh^2(4x) \cdot \operatorname{sech}^2(4x) \cdot 4 \\ &= 5 \operatorname{sech}(4x) - 20x \operatorname{sech}(4x) \tanh(4x) - 252 \tanh^2(4x) \operatorname{sech}^2(4x) \end{aligned}$$

5. (a) *Use the substitution $x = \cosh u$, $u > 0$ to show that*

$$\int \frac{1}{\sqrt{x^2-1}}\, dx = \cosh^{-1}(x) + C$$

for $x > 1$.
As in the previous integration problem, if $x = \cosh u$, then $dx = \sinh u\, du$ and

$$\begin{aligned} \int \frac{1}{\sqrt{x^2-1}}\, dx &= \int \frac{1}{\sqrt{\sinh^2 u}} \cosh u\, du \\ &= \int 1\, du = u + C = \cosh^{-1} x + C \end{aligned}$$

(b) *Use the substitution $x = \sec u$, $0 < u < \frac{\pi}{2}$, to show that*

$$\int \frac{1}{\sqrt{x^2-1}}\, dx = \ln\left|x + \sqrt{x^2-1}\right| + C$$

for $x > 1$.
Let $x = \sec u$, then $dx = \sec u \tan u\, du$ and

$$\begin{aligned} \int \frac{1}{\sqrt{x^2-1}}\, dx &= \int \frac{1}{\sqrt{\tan^2 u}} \sec u \tan u\, du \\ &= \int \sec u\, du = \ln|\sec u + \tan u| + C = \ln\left|x + \sqrt{1+x^2}\right| + C \end{aligned}$$

(c) *Use the above to show that*

$$\cosh^{-1}(x) = \ln\left|x + \sqrt{x^2-1}\right|$$

for $x > 1$.
Since both sides are antiderivatives for the same function, they differ by a constant, i.e.,

$$\cosh^{-1}(x) = \ln\left|x + \sqrt{x^2-1}\right| + k$$

We know that $\cosh^{-1}(1) = 0$ and $\ln\left|1 + \sqrt{1-1}\right| = 0$, so we have $0 = 0 + k$, or $k = 0$ so that $\cosh^{-1}(x) = \ln\left|x + \sqrt{x^2-1}\right|$.

6. *As we did in the text, find the derivative of* $\operatorname{glog}(x)$.

From the definition of $\operatorname{glog}(x)$, we know that $x \cdot \operatorname{glog}(x) = e^{\operatorname{glog}(x)}$. Differentiating both sides with respect to x gives us

$$\begin{aligned}
\operatorname{glog}(x) + x\frac{d}{dx}\operatorname{glog}(x) &= \frac{d}{dx}\operatorname{glog}(x)e^{\operatorname{glog}(x)} \\
\frac{d}{dx}\operatorname{glog}(x)\left(e^{\operatorname{glog} x} - x\right) &= -\operatorname{glog}(x) \\
\frac{d}{dx}\operatorname{glog}(x) &= -\frac{\operatorname{glog}(x)}{(e^{\operatorname{glog} x} - x)} \\
\frac{d}{dx}\operatorname{glog}(x) &= -\frac{\operatorname{glog}(x)}{x(\operatorname{glog} x - 1)}
\end{aligned}$$

7. *Find the second derivative of the Lambert W function.*

We showed that

$$W'(x) = \frac{W(x)}{x(1 + W(x))}$$

so we differentiate this to find

$$\begin{aligned}
W''(x) &= \frac{x(1 + W(x))W'(x) - W(x)(1 + W(x) + xW'(x))}{(x(1 + W(x)))^2} \\
&= \frac{xW'(x) - W(x) - W^2(x)}{(x(1 + W(x)))^2} \\
&= -\frac{W^2(x)(2 + W(x))}{(1 + W(x))^3x^2} \\
&= \frac{W(x)}{(1 + W(x))^2x^2} - \frac{W^2(x)}{x^2(1 + W(x))^3} - \frac{W(x)}{x^2(1 + W(x))}
\end{aligned}$$

8. *Using the Lambert W function, solve $xb^x = a$ for x.*

Use the defining equation for the Lambert W function:

$$x = W(y) \Longrightarrow y = xe^x.$$

So we want to get xb^x into the form ue^u. We know that $b^x = e^{x \ln b}$, so we can multiply both sides by $\ln b$ to get

$$\begin{aligned}
xb^x &= a \\
x \ln b \cdot e^{x \ln b} &= a \ln b
\end{aligned}$$

is true if and only if

$$\begin{aligned}
x \ln b &= W(a \ln b) \\
x &= \frac{W(a \ln b)}{\ln b}
\end{aligned}$$