Long Division

Long division is a collapsed version of the Euclidean Division algorithm. First, let's remind ourselves what the Euclidean Algorithm is:

Euclidean Division Algorithm: If m and n are integers and if $n > 0$, then there are unique integers q and r such that $m = nq + r$ and $0 \le r < n$.

Another way to write this is to say that $m/n = q + r/n$ with $0 \le r < n$.

Let's see how you use this to find the decimal representation of any rational number. I will illustrate the process by finding the decimal representation of $152/21$.

$152 = 21 \times 7 + 5$

This first application gives us 7, the integer part of this decimal. Each succeeding application uses 10 times the remainder from the previous step. Here are the first 6 lines, which yield a quotient of 7.238095 and a remainder of $\frac{5}{21{,}000{,}000}$.

$$152 = 21 \times 7 + 5$$

$$50 = 21 \times 2 + 8 \quad \Rightarrow \quad 5 = 21 \times \frac{2}{10} + \frac{8}{10} \quad \Rightarrow \quad 152 = 21 \times \left(7 + \frac{2}{10}\right) + \frac{8}{10}$$

$$80 = 21 \times 3 + 17 \quad \Rightarrow \quad \frac{8}{10} = 21 \times \frac{3}{10^2} + \frac{17}{10^2} \quad \Rightarrow \quad 152 = 21 \times \left(7 + \frac{2}{10} + \frac{3}{10^2}\right) + \frac{17}{10^2}$$

$$170 = 21 \times 8 + 2 \quad \Rightarrow \quad \frac{17}{10^2} = 21 \times \frac{8}{10^3} + \frac{2}{10^3} \quad \Rightarrow \quad 152 = 21 \times \left(7 + \frac{2}{10} + \frac{3}{10^2} + \frac{8}{10^3}\right) + \frac{2}{10^3}$$

$$20 = 21 \times 0 + 20 \quad \Rightarrow \quad \frac{2}{10^3} = 21 \times \frac{0}{10^4} + \frac{20}{10^4} \quad \Rightarrow \quad 152 = 21 \times \left(7 + \frac{2}{10} + \frac{3}{10^2} + \frac{8}{10^3} + \frac{0}{10^4}\right) + \frac{20}{10^4}$$

$$200 = 21 \times 9 + 11 \quad \Rightarrow \quad \frac{20}{10^4} = 21 \times \frac{9}{10^5} + \frac{11}{10^5} \quad \Rightarrow \quad 152 = 21 \times \left(7 + \frac{2}{10} + \frac{3}{10^2} + \frac{8}{10^3} + \frac{0}{10^4} + \frac{9}{10^5}\right) + \frac{11}{10^5}$$

$$110 = 21 \times 5 + 5 \quad \Rightarrow \quad \frac{11}{10^5} = 21 \times \frac{5}{10^6} + \frac{5}{10^6} \quad \Rightarrow \quad 152 = 21 \times \left(7 + \frac{2}{10} + \frac{3}{10^2} + \frac{8}{10^3} + \frac{0}{10^4} + \frac{9}{10^5} + \frac{5}{10^6}\right) + \frac{5}{10^6}$$

There are only 20 possible non-zero remainders when dividing by 21 so the cycle of the quotients that begins 238095… must repeat after at most 20 steps. In fact, since $21 = 3 \times 7$, the cycle repeats after 6 steps.

Now compare each line with the steps of the calculation from Long Division below. The final remainder 5 is equal to a remainder six steps earlier, so the cycle of quotients, 238095, will be repeated if the long division is continued.

```
       7.238095
21)152.0000000
   147
     50
     42
      80
      63
      170
      168
        20
         0
        200
        189
         110
         105
           50
```