ASSIGNMENT 1

August 28, 2006

1. Using the Trichotomy Law prove that if a and b are real numbers then one and only one of the following is possible: $a < b$, $a = b$, or $a > b$.

2. We define the *absolute value* of a real number a by

$$|a| = \begin{cases} a, & a \geq 0 \\ -a, & a \leq 0 \end{cases}$$

   Prove the following:

   (a) $|a + b| \leq |a| + |b|$.

   (b) $|xy| = |x| \cdot |y|$.

   (c) $\left|\frac{1}{x}\right| = \frac{1}{|x|}$, if $x \neq 0$.

   (d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, if $y \neq 0$.

   (e) $|x - y| \leq |x| + |y|$.

   (f) $|x| - |y| \leq |x - y|$.

3. The fact that $a^2 \geq 0$ for all real numbers a has tremendous implications. The most widely used of all inequalities is the *Schwarz inequality*:

$$x_1y_1 + x_2y_2 \leq \sqrt{x_1^2 + x_2^2}\sqrt{y_1^2 + y_2^2}$$

   Do ONE of the following:

   (a) Prove the Schwarz inequality by using $2xy \leq x^2 + y^2$ (how is this derived?) with

$$x = \frac{x_i}{\sqrt{x_1^2 + x_2^2}}, \qquad y = \frac{y_i}{\sqrt{y_1^2 + y_2^2}}$$

   first for $i = 1$ and then for $i = 2$.

   (b) Prove the Schwarz inequality by first proving that

$$(x_1^2 + x_2^2)(y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

4. Prove the following formulæ by induction

   (a) $1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$

   (b) $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$

5. Find a formula for

(a) $\sum_{i=1}^{n}(2i-1) = 1+3+5+7+\cdots+(2n-1)$

(b) $\sum_{i=1}^{n}(2i-1)^2 = 1^2+3^2+5^2+7^2+\cdots+(2n-1)^2$

Hint: What do these expressions have to do with $1+2+3+...+2n$ and $1^2+2^2+3^2+...+(2n)^2$?

6. The formula for $1^2+\cdots+n^2$ may be derived as follows. Start with the formula:

$$(k+1)^3 - k^3 = 3k^2 + 3k + 1.$$

Writing this formula for $k = 1, \ldots, n$ and adding, we get

$$\begin{array}{rcl} 2^3 - 1^3 &=& 3\cdot 1^2 + 3\cdot 1 + 1 \\ 3^3 - 2^3 &=& 3\cdot 2^2 + 3\cdot 2 + 1 \\ 4^3 - 3^3 &=& 3\cdot 3^2 + 3\cdot 3 + 1 \\ &\vdots& \\ (n+1)^3 - n^3 &=& 3\cdot n^2 + 3\cdot n + 1 \\ \hline (n+1)^3 - 1 &=& 3[1^2+\cdots n^2] + 3[1+\cdots+n] + n \end{array}$$

Solving for the first term on the right we have:

$$\begin{array}{rcl} 3[1^2+\cdots n^2] &=& (n+1)^3 - 1 - 3[1+\cdots+n] - n \\ 1^2+\cdots n^2 &=& \dfrac{n(n+1)(2n+1)}{6} \end{array}$$

Use this method to find:

(a) $1^3+2^3+3^3+4^3+\cdots+n^3$

(b) $1^4+2^4+3^4+4^4+\cdots+n^4$

(c) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \cdots + \frac{1}{n(n+1)}$