ASSIGNMENT 6

23-October-2006

1. Prove that $n^2 < 2^n$ for all $n \geq 5$. [Hint: While proving this you might be forced to prove another inequality about 2^n by a separate induction.]

2. Consider the sequence
$$0, 1, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 1, \ldots$$
where each string of zeros has one more zero than the previous. Does this sequence converge or diverge? If it converges to a limit L, prove that it converges to L. If it diverges, prove that it diverges.

3. Suppose $f\colon \mathbb{R} \to \mathbb{R}$ and $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

   (a) Prove that $f(0) = 0$.

   (b) Prove by induction that $f(nx) = nf(x)$ for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.
   Let $\alpha = f(1)$.

   (c) Prove that $f(x) = \alpha x$ for all $x \in \mathbb{N}$.

   (d) Prove that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Conclude that $f(x) = \alpha x$ for all $x \in \mathbb{Z}$.

   (e) Prove that $f(\frac{x}{n}) = \frac{f(x)}{n}$ for all $x \in \mathbb{R}$. Conclude that $f(x) = \alpha x$ for all $x \in \mathbb{Q}$.

   (f) Suppose in addition that f is continuous, *i.e.* that for all $a \in \mathbb{R}$, $\lim_{x \to a} f(x) = f(a)$. Prove that $f(x) = \alpha x$ for all $x \in \mathbb{R}$. [Remark: You have proved that the only continuous homomorphisms of the additive group $(\mathbb{R}, +)$ are of the form $f(x) = \alpha x$.]