MATH 6101 090 Solutions Assignment 7

1. *Show that if $f:\mathbb{Q}\to\mathbb{R}$ has the properties that $f(0) = 0$ and $f(x + y) = f(x) + f(y)$ for $x\in\mathbb{Q}$, then there is a number c such that $f(x)=cx$ for all $x\in\mathbb{Q}$.*
   Since $f(x + y) = f(x) + f(y)$, then $f(2) = f(1+1)=f(1) + f(1) = 2f(1)$, and then by induction, $f(n) = nf(1)$ for all integers n. Let $f(1) = c$. We then have that for every integer n, $f(n) = cn$. If $x\in\mathbb{Q}$ then $x = p/q$ for integers p and q. Thus, from the above properties, we have that $f(p/q) = pf(1/q)$. Thus, we only need to find $f(1/q)$.

   Now, $c = f(1) = f(q/q) = qf(1/q) \quad\Rightarrow\quad f(1/q) = c/q = c(1/q)$.
   This means then that $f(x) = f(p/q) = p\,f(1/q) = pc/q = c(p/q) = cx$.

2. *Suppose that $\{a_n\}$ is a sequence such that $|a_n - a_m| \le 1/|m - n|$ for all indices m, n. Prove that $a_0 = a_1 = a_2 = a_3 = \ldots = a_n = \ldots$.*
   Fix m and let $\varepsilon > 0$. For every $n \ge \max\{2/\varepsilon, 2m\}$ we know that
   $$|a_n - a_m| \le 1/|m - n| \le n/2 \le \varepsilon.$$
   Therefore $\{a_n\} \to a_m$. Since the limit of a sequence is unique and the sequence converges to each term, we must have that all the terms are equal.

3. *Suppose that $\{a_n\} \to A$ and $b_n = (a_n + a_{n+1})/3$ for all n. Does $\{b_n\}$ converge or diverge?*
   By the additive property of limits, $\{b_n\} \to 2A/3$.

4. *Let a be a fixed non-negative real number. Prove that $\lim_{n\to\infty}\sqrt[n]{a} = 1$.*
   Assume that $a > 1$. Then we can factor
   $$a - 1 = (a^{1/n} - 1)(a^{(n-1)/n} + a^{(n-2)/n} + \cdots + a^{1/n} + 1)$$
   by the Binomial Theorem. Since a > 1, we have that a – 1 > 0 and all the terms in the second factor above are positive. Therefore,
   $$0 < \frac{a-1}{a^{(n-1)/n} + a^{(n-2)/n} + \cdots + a^{1/n} + 1} = a^{1/n} - 1$$
   Now, each term in the denominator must be at least 1, which then gives us that
   $$0 < a^{1/n} - 1 = \frac{a-1}{a^{(n-1)/n} + a^{(n-2)/n} + \cdots + a^{1/n} + 1} < \frac{a-1}{\underbrace{1+1+\cdots+1+1}_{n\text{ times}}} = \frac{a-1}{n}$$
   Then by the Squeeze Theorem and the algebra of limits, we have

$$0 < a^{1/n} - 1 < \frac{a-1}{n}$$

$$\lim_{n\to\infty} 0 < \lim_{n\to\infty}\left(a^{1/n} - 1\right) < \lim_{n\to\infty}\frac{a-1}{n}$$

$$0 < \lim_{n\to\infty}\left(a^{1/n} - 1\right) < \lim_{n\to\infty}\frac{a}{n} - \lim_{n\to\infty}\frac{1}{n}$$

$$0 < \lim_{n\to\infty}\left(a^{1/n} - 1\right) < a\lim_{n\to\infty}\frac{1}{n} - \lim_{n\to\infty}\frac{1}{n}$$

$$0 < \lim_{n\to\infty}\left(a^{1/n} - 1\right) < 0$$

Therefore, $\lim_{n\to\infty}\sqrt[n]{a} = 1$. If $0 < a < 1$, then $1/a > 1$ and the above result applies, again giving $\lim_{n\to\infty}\sqrt[n]{a} = 1$.

5. *Use the Binomial Theorem to prove that if* $0 < a < 1$*, then* $\lim_{n\to\infty} na^n = 0$.

   Since $0 < a < 1$ we have that $1/a > 1$. Let $h = 1/a - 1$ and we get that $a = \frac{1}{1+h}$ where $h > 0$.

$$a^n = \frac{1}{(1+h)^n} = \frac{1}{1 + nh + \binom{n}{2}h^2 + \ldots + h^n}$$

$$na^n = \frac{n}{1 + nh + \binom{n}{2}h^2 + \ldots + h^n}$$

$$= \frac{1}{\frac{1}{n} + h + \frac{n-1}{2}h^2 + \frac{(n-1)(n-2)}{6}h^2 + \ldots + \frac{h^n}{n}}$$

Therefore,

$$\lim_{n\to\infty} na^n = \lim_{n\to\infty}\left(\frac{1}{\frac{1}{n} + h + \frac{n-1}{2}h^2 + \frac{(n-1)(n-2)}{6}h^2 + \ldots + \frac{h^n}{n}}\right) = 0.$$

6. *Prove that .*

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{mx}{m+n}\right) = 0$$

whereas

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{mx}{m+n}\right) = x$$

Let's do the second one first.

$$\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{mx}{m+n}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}\frac{mx}{m}\right)=\lim_{n\to\infty}\left(\lim_{m\to\infty}x\right)=x\ .$$

Now,

$$\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{mx}{m+n}\right)=\lim_{m\to\infty}\left(\lim_{n\to\infty}\frac{mx}{n}\right)=\lim_{m\to\infty}(0)=0.$$

7. *Suppose that $a_{n+1} = (a_n - 1)/2$ for $n = 0,1,2,\ldots.$ Prove that $\{a_n\} \to -1$.*

If $a_{n+1} = (a_n - 1)/2$, then

$$\begin{aligned}\lim_{n\to\infty}a_{n+1}&=\lim_{n\to\infty}\frac{a_n-1}{2}\\ L&=\frac{\lim_{n\to\infty}a_n-1}{2}\\ L&=\frac{L-1}{2}\\ 2L&=L-1\\ L&=-1\end{aligned}$$