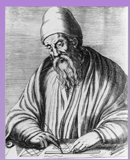
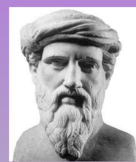




MATH 6101

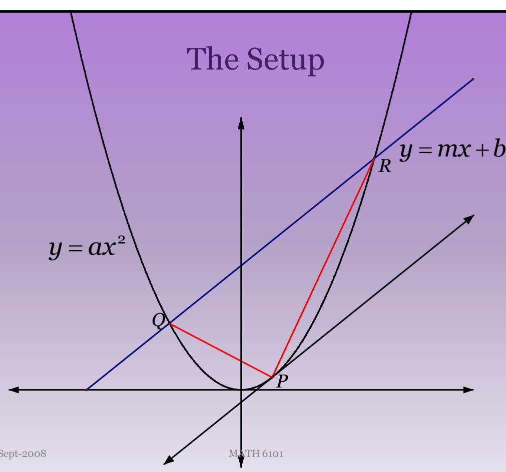
Archimedes and Quadrature

The good Christian should beware of mathematicians, and all those who make empty prophecies. The danger already exists that the mathematicians have made a covenant with the Devil to darken the spirit and to confine man in the bonds of Hell.

St. Augustine

DeGenesi ad Litteram, Book II, xviii, 37

The Setup



The Setup

P is the point at which the tangent line to the curve is parallel to the secant QR.

Where does the line intersect the parabola?

$$ax^2 = mx + b$$

$$ax^2 - mx - b = 0$$

$$x_1 = \frac{m + \sqrt{m^2 + 4ab}}{2a}, x_2 = \frac{m - \sqrt{m^2 + 4ab}}{2a}$$

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Points of Intersection

Now, we can find the points of intersection of the line and the parabola, Q and R.

$$Q = (x_2, y_2) \qquad y_2 = ax_2^2 = \frac{m^2 - m\sqrt{m^2 + 4ab} + 2ab}{2a}$$

$$Q = \left(\frac{m - \sqrt{m^2 + 4ab}}{2a}, \frac{m^2 - m\sqrt{m^2 + 4ab} + 2ab}{2a} \right)$$

$$R = (x_1, y_1) \qquad y_1 = ax_1^2 = \frac{m^2 + m\sqrt{m^2 + 4ab} + 2ab}{2a}$$

$$R = \left(\frac{m + \sqrt{m^2 + 4ab}}{2a}, \frac{m^2 + m\sqrt{m^2 + 4ab} + 2ab}{2a} \right)$$

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Slope of the Tangent Line

Here we will use some Calculus to help us.

The slope of the tangent line is the derivative of the function at the point.

$$\frac{d}{dx}(ax^2) = 2ax = m$$

$$x = \frac{m}{2a}$$

$$P = (p_1, p_2) = (x, ax^2) = \left(\frac{m}{2a}, \frac{m^2}{4a} \right)$$

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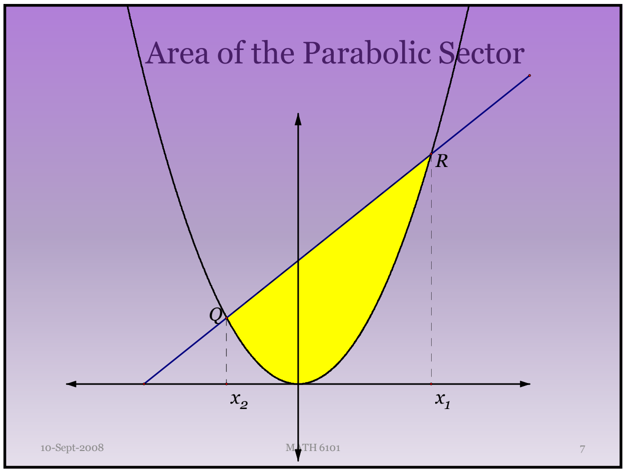


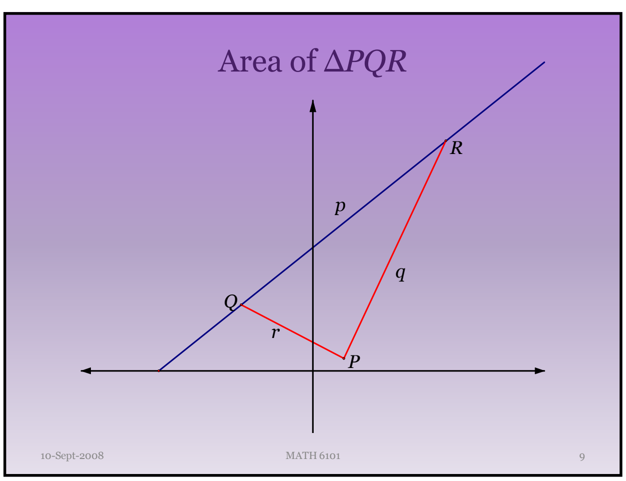
Calculus Again to the Rescue

Again, Calculus will help us find the area, *A*.

$$A = \int_{x_2}^{x_1} \left(ax^2 - (mx + b) \right) dx$$

$$= \frac{a}{3}x^3 - \frac{m}{2}x^2 + bx \Big|_{x_2}^{x_1}$$

$$A = \frac{(m^2 + 4ab)^{3/2}}{6a^2}$$



Area of Triangle

It does not look like we can find a usable angle here.

What are our options?

(1) Drop a perpendicular from P to QR and then use dot products to compute angles and areas.

(2) Drop a perpendicular from Q to PR and follow the above prescription.

(3) Drop a perpendicular from R to PQ and follow the above prescription.

(4) Use Heron's Formula.

Area of the Triangle

Using Heron's Formula:

$$p = d(Q,R) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$p = \frac{\sqrt{(m^2 + 4ab)(1 + m^2)}}{a}$$

$$q = d(P,R) = \sqrt{(x_1 - p_1)^2 + (y_1 - p_2)^2}$$

$$q = \frac{\sqrt{(m^2 + 4ab)(4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab})}}{4a}$$

Area of the Triangle

$$r = d(P,Q) = \sqrt{(p_1 - x_2)^2 + (p_2 - y_2)^2}$$

$$r = \frac{\sqrt{(m^2 + 4ab)(4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab})}}{4a}$$

Now, the semiperimeter is: $s = \dfrac{p + q + r}{2}$

$$s = \frac{\sqrt{m^2 + 4ab}}{8a}\Big(4\sqrt{1 + m^2} + \sqrt{4ab + 4 + 5m^2 + 4m\sqrt{m^2 + 4ab}} + \sqrt{4ab + 4 + 5m^2 - 4m\sqrt{m^2 + 4ab}}\Big)$$

Area of the Triangle

Uh – oh!!!!

Are we in trouble? Heron's Formula states that the area is the following product:

$$K = \sqrt{s(s-p)(s-q)(s-r)}$$

This does not look promising!!

$$\begin{aligned}K = \Bigg(\frac{(m^2+4ab)^2}{4096a^4} \Big(& 4\sqrt{1+m^2} + \sqrt{4ab+4+5m^2+4m\sqrt{m^2+4ab}} \\ & + \sqrt{4ab+4+5m^2-4m\sqrt{m^2+4ab}} \Big) \times \Big(-4\sqrt{1+m^2} \\ & + \sqrt{4ab+4+5m^2+4m\sqrt{m^2+4ab}} + \sqrt{4ab+4+5m^2-4m\sqrt{m^2+4ab}} \Big) \times \\ & \Big(4\sqrt{1+m^2} - \sqrt{4ab+4+5m^2+4m\sqrt{m^2+4ab}} \\ & + \sqrt{4ab+4+5m^2-4m\sqrt{m^2+4ab}} \Big) \times \Big(4\sqrt{1+m^2} \\ & + \sqrt{4ab+4+5m^2+4m\sqrt{m^2+4ab}} - \sqrt{4ab+4+5m^2-4m\sqrt{m^2+4ab}} \Big) \Bigg)^{1/2}\end{aligned}$$

and then a miracle occurs …

$$K = \sqrt{\frac{(m^2+4ab)^3}{64a^4}}$$

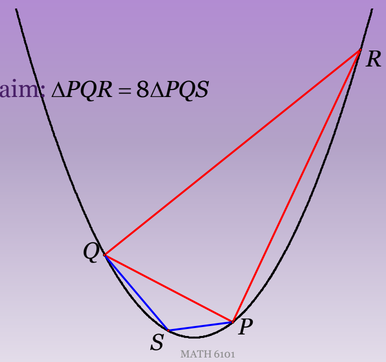
$$K = \frac{(m^2+4ab)^{3/2}}{8a^2}$$

Note then that:

$$\frac{(m^2+4ab)^{3/2}}{6a^2} = A = \frac{4}{3}K$$

How did Archimedes do this?

Claim: $\Delta PQR = 8\Delta PQS$



How did Archimedes do this?

Claim: $\Delta PQR = 8\Delta PQS$

What do we mean by "equals" here?

What did Archimedes mean by "equals"?

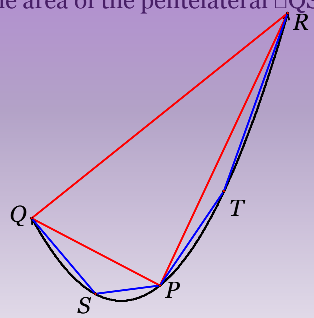
What good does this do?

What is the area of the quadrilateral □QSPR?

$$A = K + \frac{1}{8}K$$

A better approximation

What is the area of the pentelateral □QSPTR?

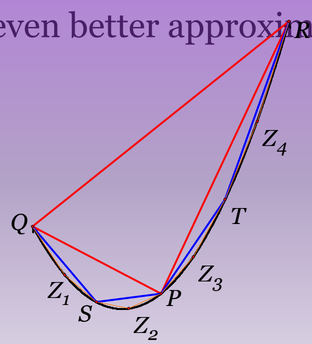


The better approximation

Note that the triangle ΔPTR is exactly the same as ΔQSP so we have that

$$A_1 = K + \frac{1}{8}K + \frac{1}{8}K = K + \frac{1}{4}K$$

An even better approximation



The next approximation

Let's go to the next level and add the four triangles given by secant lines *QS*, *SP*, *PT*, and *TR*.

$$\text{area}(\Delta QZ_1S) = \text{area}(\Delta SZ_2P) = \frac{1}{8}\text{area}(\Delta QSP)$$

$$= \frac{1}{8}\left(\frac{1}{8}K\right) = \frac{1}{64}K$$

$$\text{area}(\Delta PZ_3T) = \text{area}(\Delta TZ_4R) = \frac{1}{8}\text{area}(\Delta PTR)$$

$$= \frac{1}{8}\left(\frac{1}{8}K\right) = \frac{1}{64}K$$

The next approximation

What is the area of this new polygon that is a much better approximation to the area of the sector of the parabola?

$$A_2 = A_1 + \frac{4}{64}K = K + \frac{1}{4}K + \frac{1}{16}K$$

The next approximation

What is the area of each triangle in terms of the original stage?

$$K_3 = \frac{1}{8}K_2 = \frac{1}{8}\left(\frac{1}{8}K_1\right) = \frac{1}{8}\left(\frac{1}{8}\left(\frac{1}{8}K\right)\right) = \frac{K}{8^3}$$

What is the area of the new approximation?

$$A_3 = A_2 + 8K_3 = K + \frac{1}{4}K + \frac{1}{16}K + \frac{1}{64}K$$

The next approximation

Okay, we have a pattern to follow now.

How many triangles to we add at the next stage?

8

What is the area of each triangle in terms of the previous stage?

$$K_3 = \frac{1}{8} K_2$$

The next approximation

What is the area of the next stage?

We add twice as many triangles each of which has an eighth of the area of the previous triangle. Thus we see that in general,

$$A_n = K + \frac{1}{4}K + \frac{1}{16}K + \cdots + \frac{1}{4^n}K$$

This, too, Archimedes had found without the aid of modern algebraic notation.

The Final Analysis

Now, Archimedes has to convince his readers that "by exhaustion" this "infinite series" converges to the area of the sector of the parabola.

Now, he had to sum up the series. He knew

$$1 + \frac{1}{4} + \frac{1}{16} + \cdots + \frac{1}{4^n} + \ldots = \frac{1}{1 - \frac{1}{4}} = \frac{4}{3}$$

The Final Analysis

Therefore, Archimedes arrives at the result

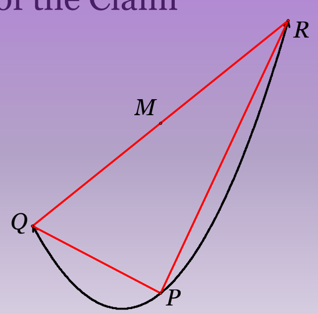
$$A = \frac{4}{3}K$$

Note that this is what we found by Calculus.

Do you think that this means that Archimedes knew the "basics" of calculus?

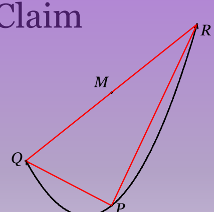
Proof of the Claim

Let M be the midpoint of QR.
Claim 1: The x-coordinate of M is the same as that of P, the vertex.



Proof of the Claim

From our coordinate geometry we find that the x-coordinate of M is given by:



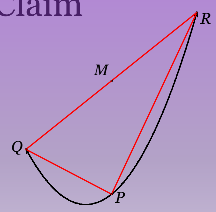
$$M_x = \frac{x_1 + x_2}{2}$$

$$= \frac{1}{2}\left(\frac{m+\sqrt{m^2+4ab}}{2a} + \frac{m-\sqrt{m^2+4ab}}{2a}\right)$$

$$= \frac{m}{2a}$$

Proof of the Claim

Claim 2: If P is a vertex and M is any point on the chord, then the ratio QM^2/PM is independent of M.



$$M_x = \frac{m}{2a} \quad M_y = \frac{y_1 + y_2}{2} = \frac{m(x_1 + x_2) + 2b}{2} = \frac{m^2}{2a} + b$$

Proof of the Claim

Now the square of the length QM is

$$(2QM)^2 = QR^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 = (x_1 - x_2)^2 + m^2(x_1 - x_2)^2$$

$$= (x_1 - x_2)^2(1 + m^2) = \left(2\frac{\sqrt{m^2 + 4ab}}{2a}\right)^2 (1 + m^2)$$

$$QM^2 = \frac{(m^2 + 4ab)(1 + m^2)}{4a^2}$$

Proof of the Claim

Now length PM is

$$PM = M_y - P_y = \frac{m^2}{2a} + b - \frac{m^2}{4a}$$

$$= \frac{(m^2 + 4ab)}{4a}$$

$$\frac{QM^2}{PM} = \frac{\frac{(m^2 + 4ab)(m^2 + 1)}{4a^2}}{\frac{(m^2 + 4ab)}{4a}} = \frac{(m^2 + 1)}{a}$$

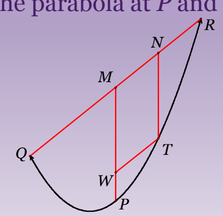
Proof of the Claim

This ratio depends only on the slope of the line and the coefficient of the parabola, not on anything else. Thus it is independent of the point along the chord.

Proof of the Claim

Claim 3: If QR is a chord of a parabola, M its midpoint and N the midpoint of MR. Drop perpendiculars from M and N to the x-axis and let them intersect the parabola at P and T. Then

$$PM = \frac{4}{3}NT$$



Proof of the Claim

Construct TW parallel to MN. Now from the previous claim we have:

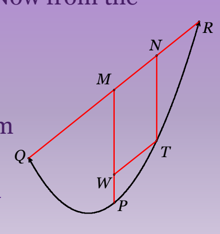
$$\frac{QM^2}{PM} = \frac{TW^2}{PW}$$

Since MNTW is a parallelogram

$$QM = 2MN = 2TW$$

$$QM^2 = 4TW^2 \Rightarrow PM = 4PW$$

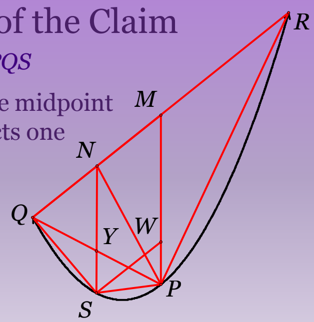
$$PV = PW + WM \Rightarrow WM = 3PW \Rightarrow PV = \frac{4}{3}WM = \frac{4}{3}NT$$



Proof of the Claim

Claim 3: $\Delta PQR = 8\Delta PQS$

We know that Y is the midpoint of PQ. Thus SY bisects one side of ΔQPM and is parallel to PM. Thus ΔQYN and ΔQPM are similar. Also, SY intersects QM at its midpoint and



Proof of the Claim

$$YN = \frac{1}{2}PV = \frac{1}{2}\frac{4}{3}SN = \frac{2}{3}SN$$

$$YN = 2SY$$

$$\Delta PQN = \Delta PYN + \Delta QYN = 2\Delta PYS + 2\Delta QYS = 2\Delta PQS$$

$$\Delta PQR = 2\Delta PQM = 4\Delta PQN = 8\Delta PQS$$

Q.E.D. Quod erat demonstrandum

Quit, enough done.