


**MATH 6101**  
**Fall 2008**

Calculus *a la* Newton




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**The Binomial Theorem**

The Binomial Theorem states that if  $n$  is a positive integer then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \dots + \binom{n}{k}x^{n-k}y^k + \dots + \binom{n}{n}y^n$$

$$= \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$$

where  $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\dots(n-k+1)}{k(k-1)(k-2)\dots 1}$

is the binomial coefficient.

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**The Binomial Theorem**

This coefficient can be computed recursively by

$$\binom{n}{0} = 1, \binom{n}{1} = n, \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \text{ for } n > k > 0$$

which is the basis for Pascal's Triangle.

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## Newton's Quest

Newton was looking for a fractional version of the Binomial Theorem. He wanted to expand:

$$y = (1 \pm x^2)^{m/n}$$

Why? He wanted to be able to find the area under these curves and he knew already how to deal with

$$y = (1 \pm x)^{m/n}$$

by the substitution  $u = 1 \pm x$   
and then the power Rule.

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What happens when  $n = 1$ ? The coefficient of  $x^{2k}$  is

$$\frac{m-0}{1} \cdot \frac{m-1}{2} \cdot \frac{m-2}{3} \cdot \frac{m-3}{4} \cdots \frac{m-(k-1)}{k}$$

We get each coefficient from the previous one by an appropriate multiplication:

$$\binom{m}{k+1} = \binom{m}{k} \cdot \frac{m-k}{k+1}$$

Would this pattern extend to fractional exponents?

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Newton "guessed" that the first coefficient should be 1 and then the subsequent coefficients satisfy:

$$\binom{m/n}{k+1} = \binom{m/n}{k} \cdot \frac{m/n-k}{k+1}$$

What would this mean for a "simple" first case?

$$(1+x^2)^{1/2}$$

$$\binom{1/2}{0} = 1$$

$$\binom{1/2}{1} = \binom{1/2}{0} \frac{1/2-0}{1} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

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$$\binom{1/2}{2} = \binom{1/2}{1} \frac{1/2-1}{2} = \frac{1}{2} \left( -\frac{1}{4} \right) = -\frac{1}{8}$$

$$\binom{1/2}{3} = \binom{1/2}{2} \frac{1/2-2}{3} = -\frac{1}{8} \left( -\frac{1}{2} \right) = \frac{1}{16}$$

$$\binom{1/2}{4} = \binom{1/2}{3} \frac{1/2-3}{4} = \frac{1}{16} \left( -\frac{5}{8} \right) = -\frac{5}{128}$$

$$\binom{1/2}{5} = \binom{1/2}{4} \frac{1/2-4}{5} = -\frac{5}{128} \left( -\frac{7}{10} \right) = \frac{7}{256}$$

All of this gives

$$(1+x^2)^{1/2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \frac{7}{256}x^{10} + \dots$$

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Substituting  $x$  for  $x^2$  gives

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 + \dots$$

Substituting  $-x^2$  for  $x^2$  gives

$$(1-x^2)^{1/2} = 1 + \frac{1}{2}(-x^2) - \frac{1}{8}(-x^2)^2 + \frac{1}{16}(-x^2)^3 - \frac{5}{128}(-x^2)^4 + \dots$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \dots$$

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### The Fractional Binomial Theorem

For every rational number  $r$  and non-negative integer  $k$ , recursively define the coefficient

$$\binom{r}{0} = 1, \binom{r}{k+1} = \binom{r}{k} \cdot \frac{r-k}{k+1} \quad k = 0, 1, 2, 3, \dots$$

then

$$(1+x)^r = 1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \binom{r}{4}x^4 + \dots \quad \text{for } |x| < 1.$$

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### Newton's Justification

**Multiply it out!**

$$\begin{array}{r}
 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \dots \\
 \times \\
 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \dots \\
 \hline
 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \dots \\
 - \frac{1}{2}x^2 + \frac{1}{4}x^4 + \frac{1}{16}x^6 + \frac{1}{32}x^8 + \dots \\
 \hline
 -\frac{1}{8}x^4 + \frac{1}{16}x^6 + \frac{1}{64}x^8 + \dots = 1 - x^2 \\
 \quad -\frac{1}{16}x^6 + \frac{1}{32}x^8 + \dots \\
 \quad \quad -\frac{5}{128}x^8 + \dots
 \end{array}$$

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### Newton's Results

**Likewise:**

$$\begin{aligned}
 (1-x^2)^{1/3} &= 1 - \frac{1}{3}x^2 - \frac{1}{9}x^4 - \frac{5}{81}x^6 - \frac{10}{243}x^8 - \dots \\
 (1-x^2)^{2/3} &= 1 - \frac{2}{3}x^2 - \frac{1}{9}x^4 - \frac{4}{81}x^6 - \frac{7}{243}x^8 - \dots \\
 (1-x^2)^{1/4} &= 1 - \frac{1}{4}x^2 - \frac{3}{32}x^4 - \frac{7}{128}x^6 - \frac{77}{2048}x^8 - \dots \\
 (1-x^2)^{3/4} &= 1 - \frac{3}{4}x^2 - \frac{3}{32}x^4 - \frac{5}{128}x^6 - \frac{45}{2048}x^8 - \dots
 \end{aligned}$$

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### Newton's Results

**What is more this works for negative exponents!!**

$$\begin{aligned}
 (1+x)^{-1} &= 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \binom{-1}{3}x^3 + \binom{-1}{4}x^4 + \dots \\
 &= 1 + \frac{-1}{1}x + \frac{(-1)(-2)}{1 \cdot 2}x^2 + \frac{(-1)(-2)(-3)}{1 \cdot 2 \cdot 3}x^3 \\
 &\quad + \frac{(-1)(-2)(-3)(-4)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 + \dots \\
 &= 1 - x + x^2 - x^3 + x^4 - \dots
 \end{aligned}$$

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### Newton's Results

Replace  $x$  by  $-x$  and you get

$$(1-x)^{-1} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \dots$$

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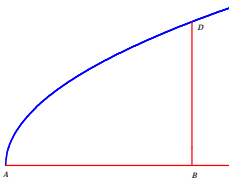
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### Areas and Infinite Series

Let  $A$  be the origin,  $AB = x$   
and  $BD = y$

**Rule I:** If  $ax^{m/n} = y$ , then

$$\text{area}(ABD) = \frac{an}{m+n} x^{(m+n)/n}$$


Translation:

$$\int x^r dx = \frac{1}{r+1} x^{r+1} + C \quad r \neq -1$$

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### Areas and Infinite Series

**Rule II:** If the Value of  $y$  be made up of several such Terms, the Area likewise shall be made up of the Areas which result from every one of the Terms.

Translation:

$$\int (af(x) + bg(x)) dx = a \int f(x) dx + b \int g(x) dx$$

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## Areas and Infinite Series

**Rule III:** But if the value of  $y$ , or any of it's Terms be more compounded than the foregoing, it must be reduced into more simple Terms; be performing the Operation in Letters, after the same Manner as Arithmeticians divide in Decimal Numbers, extract the Square Root, or resolve affected Equations.

Translation: Use long division to write

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \dots$$

then integrate

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## Newton's Example

Consider the hyperbola

$$y = \frac{a^2}{b+x}$$

By long division we get

$$y = \frac{a^2}{b+x} = \frac{a^2}{b} - \frac{a^2 x}{b^2} + \frac{a^2 x^2}{b^3} - \frac{a^2 x^3}{b^4} \dots$$

Then by Rules I and II, the area is given by

$$\frac{a^2 x}{b} - \frac{a^2 x^2}{2b^2} + \frac{a^2 x^3}{3b^3} - \frac{a^2 x^4}{4b^4} \dots$$

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## Newton's Next Example

Similar to the previous

$$y = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \dots$$

Then by Rules I and II, the area from the origin to the point  $x$  is given by

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 \dots$$

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### Newton's Next Example

Newton was able to recognize this same area as the arctangent:

$$\int \frac{1}{1+x^2} dx = \arctan(x) + C$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 \dots$$

This had been discovered by James Gregory.

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