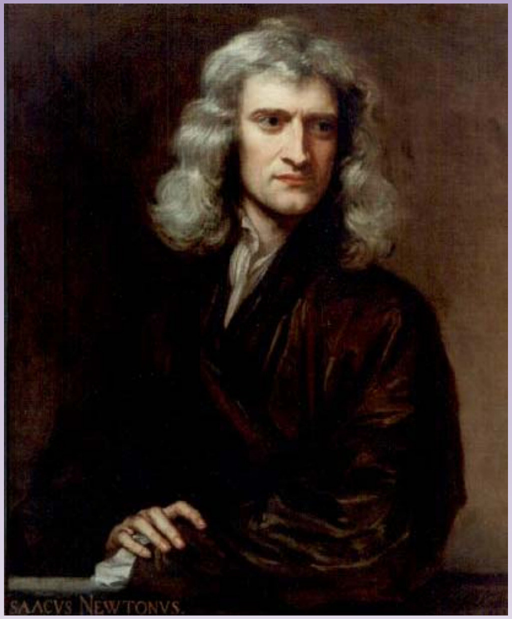
MATH 6101
Fall 2008

Calculus *a la* Newton



The Binomial Theorem

The Binomial Theorem states that if n is a positive integer then

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{k}x^{n-k}y^k + \cdots + \binom{n}{n}y^n$$

$$= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 1}$

is the binomial coefficient.

The Binomial Theorem

This coefficient can be computed recursively by

$$\binom{n}{0} = 1, \quad \binom{n}{1} = n \quad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \quad \text{for } n > k > 0$$

which is the basis for Pascal's Triangle.

Newton's Quest

Newton was looking for a fractional version of the Binomial Theorem. He wanted to expand:

$$y = \left(1 \pm x^2\right)^{m/n}$$

Why? He wanted to be able to find the area under these curves and he knew already how to deal with

$$y = \left(1 \pm x\right)^{m/n}$$

by the substitution $u = 1 \pm x$

and then the power Rule.

What happens when $n = 1$? The coefficient of x^{2k} is

$$\frac{m-0}{1} \bullet \frac{m-1}{2} \bullet \frac{m-2}{3} \bullet \frac{m-3}{4} \bullet \cdots \bullet \frac{m-(k-1)}{k}$$

We get each coefficient from the previous one by an appropriate multiplication:

$$\binom{m}{k+1} = \binom{m}{k} \bullet \frac{m-k}{k+1}$$

Would this pattern extend to fractional exponents?

Newton "guessed" that the first coefficient should be 1 and then the subsequent coefficients satisfy:

$$\binom{m/n}{k+1} = \binom{m/n}{k} \bullet \frac{m/n-k}{k+1}$$

What would this mean for a "simple" first case?

$$\left(1+x^2\right)^{1/2}$$

$$\binom{1/2}{0} = 1$$

$$\binom{1/2}{1} = \binom{1/2}{0}\frac{1/2-0}{1} = 1 \cdot \frac{1}{2} = \frac{1}{2}$$

$$\binom{1/2}{2} = \binom{1/2}{1}\frac{1/2-1}{2} = \frac{1}{2}\left(-\frac{1}{4}\right) = -\frac{1}{8}$$

$$\binom{1/2}{3} = \binom{1/2}{2}\frac{1/2-2}{3} = -\frac{1}{8}\left(-\frac{1}{2}\right) = \frac{1}{16}$$

$$\binom{1/2}{4} = \binom{1/2}{3}\frac{1/2-3}{4} = \frac{1}{16}\left(-\frac{5}{8}\right) = -\frac{5}{128}$$

$$\binom{1/2}{5} = \binom{1/2}{4}\frac{1/2-4}{5} = -\frac{5}{128}\left(-\frac{7}{10}\right) = \frac{7}{256}$$

All of this gives

$$\left(1+x^2\right)^{1/2} = 1 + \frac{1}{2}x^2 - \frac{1}{8}x^4 + \frac{1}{16}x^6 - \frac{5}{128}x^8 + \frac{7}{256}x^{10} + \cdots$$

Substituting x for x^2 gives

$$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 + \cdots$$

Substituting $-x^2$ for x^2 gives

$$(1-x^2)^{1/2} = 1 + \frac{1}{2}(-x^2) - \frac{1}{8}(-x^2)^2 + \frac{1}{16}(-x^2)^3 - \frac{5}{128}(-x^2)^4 + \cdots$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \cdots$$

The Fractional Binomial Theorem

For every rational number r and non-negative integer k, recursively define the coefficient

$$\binom{r}{0} = 1, \binom{r}{k+1} = \binom{r}{k} \bullet \frac{r-k}{k+1} \quad k = 0,1,2,3,\ldots$$

then

$$(1+x)^r = 1 + \binom{r}{1}x + \binom{r}{2}x^2 + \binom{r}{3}x^3 + \binom{r}{4}x^4 + \cdots \quad \text{for } |x| < 1.$$

Newton's Justification

Multiply it out!

$$1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^2-\cdots$$

$\times$

$$1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^2-\cdots$$

$$\left.\begin{aligned}
1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8-\cdots\\
-\frac{1}{2}x^2+\frac{1}{4}x^4+\frac{1}{16}x^6+\frac{1}{32}x^8+\cdots\\
-\frac{1}{8}x^4+\frac{1}{16}x^6+\frac{1}{64}x^8+\cdots\\
-\frac{1}{16}x^6+\frac{1}{32}x^8+\cdots\\
-\frac{5}{128}x^8+\cdots
\end{aligned}\right\}=1-x^2$$

Newton's Results

Likewise:

$$\left(1-x^2\right)^{1/3}=1-\frac{1}{3}x^2-\frac{1}{9}x^4-\frac{5}{81}x^6-\frac{10}{243}x^8-\cdots$$

$$\left(1-x^2\right)^{2/3}=1-\frac{2}{3}x^2-\frac{1}{9}x^4-\frac{4}{81}x^6-\frac{7}{243}x^8-\cdots$$

$$\left(1-x^2\right)^{1/4}=1-\frac{1}{4}x^2-\frac{3}{32}x^4-\frac{7}{128}x^6-\frac{77}{2048}x^8-\cdots$$

$$\left(1-x^2\right)^{3/4}=1-\frac{3}{4}x^2-\frac{3}{32}x^4-\frac{5}{128}x^6-\frac{45}{2048}x^8-\cdots$$

Newton's Results

What is more this works for negative exponents!!

$$(1+x)^{-1} = 1 + \binom{-1}{1}x + \binom{-1}{2}x^2 + \binom{-1}{3}x^3 + \binom{-1}{4}x^4 + \cdots$$

$$= 1 + \frac{-1}{1}x + \frac{(-1)(-2)}{1\cdot 2}x^2 + \frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3}x^3$$

$$+ \frac{(-1)(-2)(-3)(-4)}{1\cdot 2\cdot 3\cdot 4}x^4 + \cdots$$

$$= 1 - x + x^2 - x^3 + x^4 -$$

Newton's Results

Replace x by $-x$ and you get

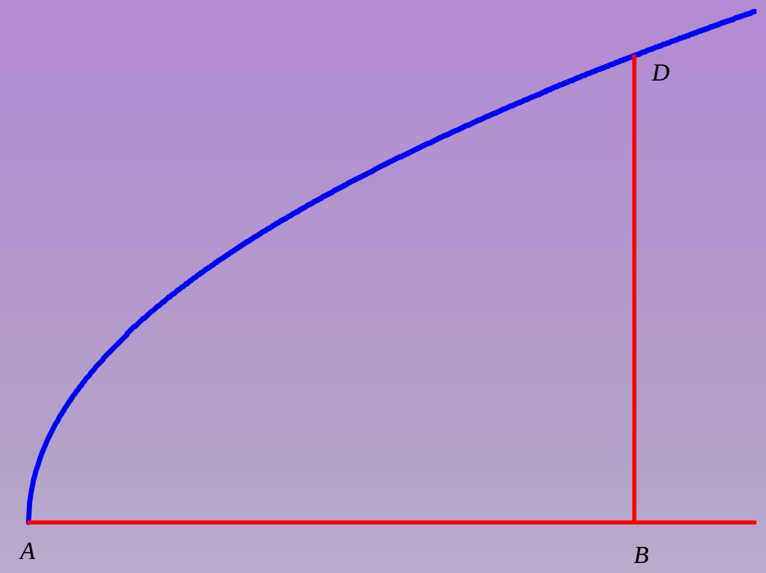
$$\left(1-x\right)^{-1}=\frac{1}{1-x}=1+x+x^2+x^3+x^4+$$

Areas and Infinite Series

Let A be the origin, $AB = x$ and $BD = y$

Rule I: If $ax^{m/n} = y$, then

$$\text{area}(ABD) = \frac{an}{m+n} x^{(m+n)/n}$$



Translation:

$$\int x^r dx = \frac{1}{r+1} x^{r+1} + C \quad r \neq -1$$

Areas and Infinite Series

Rule II: If the Value of y be made up of several such Terms, the Area likewise shall be made up of the Areas which result from every one of the Terms.

Translation:

$$\int \left(af(x)+bg(x)\right)dx = a\int f(x)dx + b\int g(x)dx$$

Areas and Infinite Series

Rule III: But if the value of *y*, or any of it's Terms be more compounded than the foregoing, it must be reduced into more simple Terms; be performing the Operation in Letters, after the same Manner as Arithmeticians divide in Decimal Numbers, extract the Square Root, or resolve affected Equations.

Translation: Use long division to write

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \cdots$$

then integrate

Newton's Example

Consider the hyperbola

$$y = \frac{a^2}{b+x}$$

By long division we get

$$y = \frac{a^2}{b+x} = \frac{a^2}{b} - \frac{a^2 x}{b^2} + \frac{a^2 x^2}{b^3} - \frac{a^2 x^3}{b^4} \cdots$$

Then by Rules I and II, the area is given by

$$\frac{a^2 x}{b} - \frac{a^2 x^2}{2b^2} + \frac{a^2 x^3}{3b^3} - \frac{a^2 x^4}{4b^4} \cdots$$

Newton's Next Example

Similar to the previous

$$y = \frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + x^8 \cdots$$

Then by Rules I and II, the area from the origin to the point x is given by

$$x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 \cdots$$

Newton's Next Example

Newton was able to recognize this same area as the arctangent:

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\arctan(x) = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \frac{1}{9}x^9 \cdots$$

This had been discovered by James Gregory.