

MATH 6101 Fall 2008

Newton and Differential Equations



A Differential Equation

What is a differential equation?

A differential equation is an equation relating the quantities x , y and y' and possibly higher derivatives of y .

Examples: $y' = x + y$

$$y'' - y = 0$$

$$(1 + x^2)y'' + 2xy' + 4x^2y = 0$$

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Differential Equations

How are differential equations used?

Newton's Second Law:

$$F = ma = \frac{d(mv)}{dt} = m \frac{d^2s}{dt^2}$$

Radioactive decay:

$$\frac{dP}{dt} = -kP$$

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Differential Equations

Newton's Law of Cooling:

$$\frac{dQ}{dt} = h \cdot A(T_0 - T_{env})$$

The wave equation:

$$\frac{\partial^2 P}{\partial t^2} = c^2 \nabla^2 u$$

The heat equation:

$$\frac{\partial u}{\partial t} - k \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) = 0$$

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Differential Equations: Modern Solution

Consider the following simple differential equation:

$$\frac{dP}{dt} = k \cdot P$$

Our "modern process" is *Separation of Variables* introduced by l'Hospital in 1750.

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Separation of Variables

$$\frac{dP}{dt} = k \cdot P$$

$$\frac{dP}{P} = k \cdot dt$$

$$\int \frac{dP}{P} = \int k \cdot dt = k \int dt$$

$$\ln(P) + C_1 = kt + C_2$$

$$\ln(P) = kt + C$$

$$P(t) = e^{kt+C} = Ae^{kt}$$

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Newton's Method of Series

$$\frac{dP}{dt} = k \cdot P$$

Now, assume that we can express P as a function of t by:

$$P(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then $\frac{dP}{dt} = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and our first equation gives

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Newton's Method of Series

$$\frac{dP}{dt} = k \cdot P$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = k \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} k a_n t^n$$

$$a_1 t^0 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \dots = k a_0 t^0 + k a_1 t + k a_2 t^2 + k a_3 t^3 + \dots$$

Setting corresponding powers of t equal, gives the following

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Newton's Method of Series

$$a_1 = k a_0$$

$$2a_2 = k a_1 \Rightarrow a_2 = \frac{k}{2} a_1 = \frac{k^2}{2} a_0$$

$$3a_3 = k a_2 \Rightarrow a_3 = \frac{k}{3} a_2 = \frac{k^3}{2 \cdot 3} a_0$$

$$4a_4 = k a_3 \Rightarrow a_4 = \frac{k}{4} a_3 = \frac{k^4}{2 \cdot 3 \cdot 4} a_0$$

$$\vdots$$

$$n a_n = k a_{n-1} \Rightarrow a_n = \frac{k}{n} a_{n-1} = \frac{k^n}{n!} a_0$$

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Newton's Method of Series

$$P(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_0 \frac{k^n}{n!} t^n = a_0 \sum_{n=0}^{\infty} \frac{(kt)^n}{n!}$$

We will show later that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore,

$$P(t) = a_0 \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} = a_0 e^{kt}$$

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Newton's Method of Series

Example 2: Find a solution to the following equation

$$y'' + y = 0$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

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Newton's Method of Series

$$y'' + y = 0$$

$$y'' = -y$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} = - \sum_{n=0}^{\infty} a_n x^n$$

$$2a_2 + 3 \cdot 2a_3 x + 4 \cdot 3a_4 x^2 + 5 \cdot 4a_5 x^3 + \dots = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - \dots$$

This gives us that

$$2a_2 = -a_0 \quad 4 \cdot 3a_4 = -a_2 \quad 6 \cdot 5a_6 = -a_4 \quad 8 \cdot 7a_8 = -a_6$$

$$3 \cdot 2a_3 = -a_1 \quad 5 \cdot 4a_5 = -a_3 \quad 7 \cdot 6a_7 = -a_5 \quad 9 \cdot 8a_9 = -a_7$$

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Newton's Method of Series

$$2a_2 = -a_0 \Rightarrow a_2 = -\frac{1}{2}a_0 = -\frac{1}{2!}a_0$$

$$3 \cdot 2a_3 = -a_1 \Rightarrow a_3 = -\frac{1}{2 \cdot 3}a_1 = -\frac{1}{3!}a_1$$

$$4 \cdot 3a_4 = -a_2 \Rightarrow a_4 = -\frac{1}{4 \cdot 3}a_2 = \frac{1}{2 \cdot 3 \cdot 4}a_0 = \frac{1}{4!}a_0$$

$$5 \cdot 4a_5 = -a_3 \Rightarrow a_5 = -\frac{1}{5 \cdot 4}a_3 = -\frac{1}{2 \cdot 3 \cdot 4 \cdot 5}a_1 = -\frac{1}{5!}a_1$$

$$6 \cdot 5a_6 = -a_4 \Rightarrow a_6 = -\frac{1}{6 \cdot 5}a_4 = -\frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}a_0 = -\frac{1}{6!}a_0$$

$$7 \cdot 6a_7 = -a_5 \Rightarrow a_7 = -\frac{1}{7 \cdot 6}a_5 = -\frac{1}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7}a_1 = -\frac{1}{7!}a_1$$

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Newton's Method of Series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$$

$$= a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4 + \frac{1}{5!} a_1 x^5 - \frac{1}{6!} a_0 x^6 - \frac{1}{7!} a_1 x^7 + \dots$$

$$= a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots \right) + a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots \right)$$

Let

$$c(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots$$

and

$$s(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots$$

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Newton's Method of Series

This gives us that

$$y = a_0 c(x) + a_1 s(x)$$

Note that:

$$\begin{aligned} [c(x)]^2 + [s(x)]^2 &= \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \dots \right)^2 + \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \dots \right)^2 \\ &= 1 - \frac{2}{2!} x^2 + \frac{1}{2!2!} x^4 + \frac{2}{4!} x^4 - \frac{2}{2!4!} x^6 - \frac{2}{6!} x^6 + x^2 - \frac{2}{3!} x^4 + \frac{1}{3!3!} x^4 + \frac{2}{5!} x^6 + \dots \\ &= 1 + (1-1)x^2 + \left(\frac{1}{2!2!} + \frac{2}{4!} - \frac{2}{2!4!} \right) x^4 + \left(-\frac{2}{2!4!} - \frac{2}{6!} + \frac{1}{3!3!} + \frac{2}{5!} \right) x^6 + \dots \\ &= 1 + 0x^2 + \left(\frac{1}{4} + \frac{1}{12} - \frac{1}{3} \right) x^4 + \left(-\frac{1}{24} - \frac{1}{360} + \frac{1}{36} + \frac{1}{60} \right) x^6 + \dots \\ &= 1 + 0x^2 + \left(\frac{4}{12} - \frac{4}{12} \right) x^4 + \left(-\frac{16}{360} + \frac{16}{360} \right) x^6 + \dots \\ &= 1 \end{aligned}$$

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Newton's Method of Series

Also note that:

$$c(0) = 1$$

$$s(0) = 0$$

Further note that:

$$\frac{d}{dx} \cos(x) = -\sin(x)$$

$$\frac{d}{dx} \sin(x) = \cos(x)$$

We can show that:

$$c(x) = \cos(x)$$

$$s(x) = \sin(x)$$

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Your Turn

Solve by Newton's method (*i.e.*, find the first 5 non-zero terms):

1) $y' = y + \frac{1}{1-x}$

2) $y' = 1 + xy$

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Solutions

$$y' = y + \frac{1}{1-x}$$

$$y(x) = a_0 + (1+a_0)x + \frac{1}{2}a_0x^2 + \frac{1}{3}\left(1 + \frac{1}{2}a_0\right)x^3 + \frac{1}{6}\left(\frac{1}{4}a_0 - 1\right)x^4 + \frac{1}{6}\left(1 + \frac{1}{20}a_0\right)x^5 + \dots$$

$$y' = 1 + xy$$

$$y(x) = a_0 + x + \frac{1}{2}a_0x^2 + \frac{1}{3}x^3 + \frac{1}{8}a_0x^4 + \frac{1}{15}x^5 + \dots$$

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Solving Algebraic Equations

How can we solve an equation of the following form?

$$y^2 + 3y - 4 - x^2y + 2x = 0$$

For this technique we do the same: assume that y has a power series expansion in terms of x .

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Where does this lead us?

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Solving Algebraic Equations

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^2 + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) - 4 - x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) + 2x = 0$$

How do we cube out a series? This seems to be the only sticking point to proceeding with this process.

We will make a couple of definitions and assumptions.

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Solving Algebraic Equations

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Given the above series define the *tail series* to be

$$t_n = \sum_{k=n}^{\infty} a_k x^k = a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \dots$$

We then have that

$$t_0 = y \quad \text{and} \quad t_n = a_n x^n + t_{n+1}$$

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Solving Algebraic Equations

Given an algebraic equation we will write it as

$$F(x, y) = 0$$

And we will begin our procedure by setting

$$F_0(x, y) = F(x, y)$$

The k th iteration of our process consists of three steps

- 1) Extract the x^k level equation from $F_k(x, t_k) = 0$. That is find the coefficients of x^k and set them equal to 0.
- 2) Solve the x^k level equation for a_k .
- 3) Substitute $t_k = a_k x^k + t_{k+1}$ into $F_k(x, t_k) = 0$ and use the Binomial Theorem to obtain the new equation

$$F_{k+1}(x, t_{k+1})$$

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Solving Algebraic Equations

Example:

$$F(x, y) = y^2 + 3y - 4 - x^2 y + 2x = 0$$

$$F_0(x, t_0) : t_0^2 + 3t_0 - 4 - x^2 t_0 + 2x = 0$$

Step 1: Since $t_0 = a_0 + a_1 x + a_2 x^2 + \dots$ and the x^0 level equation ignores all non-constant terms in $F_0(x, t_0)$:

x^0 level:

$$a_0^2 + 3a_0 - 4 = 0 \Rightarrow (a_0 + 4)(a_0 - 1) = 0 \Rightarrow a_0 = -4 \text{ or } a_0 = 1$$

This will give us two solutions and two equations

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Solving Algebraic Equations

Solution 1: $a_0 = 1$

Now, replace t_0 with $1 + t_1$ to get:

$$(1 + t_1)^2 + 3(1 + t_1) - 4 - x^2(1 + t_1) + 2x = 0$$

$$1 + 2t_1 + t_1^2 + 3 + 3t_1 - 4 - x^2 - t_1 x^2 + 2x = 0$$

$$5t_1 + t_1^2 - x^2 - t_1 x^2 + 2x = 0$$

Ignore all terms with degree > 1 .

$$5a_1 + 2 = 0$$

$$a_1 = -\frac{2}{5}$$

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Solving Algebraic Equations

$$F_1(x, t_1) = 5t_1 + t_1^2 - x^2 - t_1x^2 + 2x = 0$$

Now, replace t_1 with $-\frac{2}{5}x + t_2$

$$5t_1 + t_1^2 - x^2 - t_1x^2 + 2x = 0$$

$$5\left(-\frac{2}{5}x + t_2\right) + \left(-\frac{2}{5}x + t_2\right)^2 - x^2 - \left(-\frac{2}{5}x + t_2\right)x^2 + 2x = 0$$

$$5t_2 + t_2^2 - \frac{4}{5}xt_2 - x^2t_2 - \frac{21}{25}x^2 + \frac{2}{5}x^3 = 0$$

Ignore all terms with degree > 2 .

$$5a_2 - \frac{21}{25} = 0$$

$$a_2 = \frac{21}{125}$$

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Solving Algebraic Equations

$$F_2(x, t_2) = 5t_2 + t_2^2 - \frac{4}{5}xt_2 - x^2t_2 - \frac{21}{25}x^2 + \frac{2}{5}x^3 = 0$$

Now, replace t_2 with $\frac{21}{125}x^2 + t_3$

$$5t_3 + t_3^2 - \frac{4}{5}xt_3 - \frac{83}{125}x^2t_3 + \frac{166}{625}x^3 - \frac{2184}{15625}x^4 = 0$$

Ignore all terms with degree > 3 .

$$5a_3 + \frac{166}{625} = 0$$

$$a_3 = -\frac{166}{3125}$$

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Solving Algebraic Equations

So, we find a series expansion for y of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n = 1 - \frac{2}{5}x + \frac{21}{125}x^2 - \frac{166}{3125}x^3 + \dots$$

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Your Turn

- 1) Find the solution corresponding to $a_0 = -4$
- 2) Find the first four terms of the infinite series expansion of y if $y^3 + xy = 1$.

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Your Turn

- 1) We can solve the original equation for y and then find the series expansion for that. We should get the same answer.

$$y^2 + 3y - 4 - x^2y + 2x = 0$$

$$y^2 + (3 - x^2)y + (2x - 4) = 0$$

$$y = \frac{(x^2 - 3) \pm \sqrt{(3 - x^2)^2 - 4(2x - 4)}}{2}$$

$$y = \frac{(x^2 - 3) \pm \sqrt{x^4 - 6x^2 - 8x + 25}}{2}$$

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Your Turn

$$y_1 = \frac{x^2 - 3 + \sqrt{x^4 - 6x^2 - 8x + 25}}{2}$$

$$= 1 - \frac{2}{5}x + \frac{21}{125}x^2 - \frac{166}{3125}x^3 + \frac{304}{15625}x^4 + \dots$$

$$y_2 = \frac{x^2 - 3 - \sqrt{x^4 - 6x^2 - 8x + 25}}{2}$$

$$= -4 + \frac{2}{5}x + \frac{104}{125}x^2 + \frac{166}{3125}x^3 - \frac{304}{15625}x^4 + \dots$$

This is what we should have found with our other process.

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Your Turn

2) Find the first four terms of the infinite series expansion of y if $y^3 + xy = 1$.

$$y^3 + xy - 1 = 0$$

$$y = \frac{1}{6} \sqrt[3]{108 + 12\sqrt{12x^3 + 81}} - \frac{2x}{\sqrt[3]{108 + 12\sqrt{12x^3 + 81}}}$$

$$y = 1 - \frac{1}{3}x + \frac{1}{81}x^3 + \frac{1}{243}x^4 + \dots$$

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Problem 2

$$F(x, y) = y^3 + xy - 1 = 0$$

Stage 0: $F_0(x, t_0) = t_0^3 + xt_0 - 1 = 0$

$$a_0^3 - 1 = 0$$

$$a_0 = 1$$

Stage 1: In $F_0(x, t_0)$ replace $t_0 = 1 + t_1$

$$F_1(x, t_1) = (1 + t_1)^3 + x(1 + t_1) - 1 = 0$$

$$F_1(x, t_1) = 3t_1 + 3t_1^2 + t_1^3 + xt_1 + x = 0$$

$$3a_1 + 1 = 0$$

$$a_1 = -\frac{1}{3}$$

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Problem 2

Stage 2: In $F_1(x, t_1)$ replace $t_1 = -\frac{1}{3}x + t_2$

$$F_1(x, t_1) = 3t_1 + 3t_1^2 + t_1^3 + xt_1 + x = 0$$

$$F_2(x, t_2) = 3\left(-\frac{1}{3}x + t_2\right) + 3\left(-\frac{1}{3}x + t_2\right)^2 + \left(-\frac{1}{3}x + t_2\right)^3 + x\left(-\frac{1}{3}x + t_2\right) + x = 0$$

$$= 3t_2 + 3t_2^2 + t_2^3 - xt_2 + \frac{1}{3}x^2t_2 - xt_2^2 - \frac{1}{27}x^3 = 0$$

$$3a_2 = 0$$

$$a_2 = 0$$

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Problem 2

Stage 3: In $F_2(x, t_2)$ replace $t_2 = 0x^2 + t_3 = t_3$

$$F_2(x, t_2) = 3t_2 + 3t_2^2 + t_2^3 - xt_2 + \frac{1}{3}x^2t_2 - xt_2^2 - \frac{1}{27}x^3 = 0$$

$$F_3(x, t_3) = 3t_3 + 3t_3^2 + t_3^3 - xt_3 + \frac{1}{3}x^2t_3 - xt_3^2 - \frac{1}{27}x^3 = 0$$

$$3a_3 - \frac{1}{27} = 0$$

$$a_3 = \frac{1}{81}$$

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Problem 2

Stage 4: In $F_3(x, t_3)$ replace $t_3 = \frac{1}{81}x^3 + t_4$

$$F_3(x, t_3) = 3t_3 + 3t_3^2 + t_3^3 - xt_3 + \frac{1}{3}x^2t_3 - xt_3^2 - \frac{1}{27}x^3 = 0$$

$$F_4(x, t_4) = 3\left(\frac{x^3}{81} + t_4\right) + 3\left(\frac{x^3}{81} + t_4\right)^2 + \left(\frac{x^3}{81} + t_4\right)^3 - x\left(\frac{x^3}{81} + t_4\right) +$$

$$-\frac{1}{3}x^2\left(\frac{x^3}{81} + t_4\right) - x\left(\frac{x^3}{81} + t_4\right)^2 - \frac{1}{27}x^3 = 0$$

$$= 3t_4 + 3t_4^2 + t_4^3 - xt_4 + \frac{x^2t_4}{3} + \frac{2x^3t_4}{27} - \frac{2x^4t_4}{81} + \frac{x^6t_4}{2187} - xt_4^2 +$$

$$-\frac{x^3t_4^2}{27} - \frac{x^4}{81} + \frac{x^5}{243} + \frac{x^6}{2187} - \frac{x^7}{6561} + \frac{x^9}{531441} = 0$$

$$3a_4 - \frac{1}{81} = 0$$

$$a_4 = \frac{1}{243}$$

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