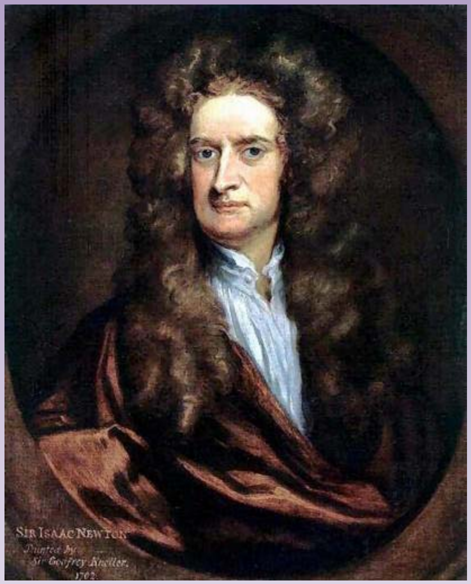
MATH 6101
Fall 2008

Newton and Differential Equations



A Differential Equation

What is a differential equation?

A differential equation is an equation relating the quantities x, y and y' and possibly higher derivatives of y.

Examples: $y' = x + y$

$$y'' - y = 0$$

$$(1 + x^2)y'' + 2xy' + 4x^2 y = 0$$

Differential Equations

How are differential equations used?

Newton's Second Law:

$$F = ma = \frac{d(mv)}{dt} = m\frac{d^2 s}{dt^2}$$

Radioactive decay:

$$\frac{dP}{dt} = -kP$$

Differential Equations

Newton's Law of Cooling:

$$\frac{dQ}{dt} = h \cdot A(T_0 - T_{env})$$

The wave equation:

$$\frac{\partial^2 P}{\partial t^2} = c^2 \nabla^2 u$$

The heat equation:

$$\frac{\partial u}{\partial t} - k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$$

Differential Equations: Modern Solution

Consider the following simple differential equation:

$$\frac{dP}{dt} = k \cdot P$$

Our "modern process" is *Separation of Variables* introduced by l'Hospital in 1750.

Separation of Variables

$$\frac{dP}{dt} = k \cdot P$$

$$\frac{dP}{P} = k \cdot dt$$

$$\int \frac{dP}{P} = \int k \cdot dt = k \int dt$$

$$\ln(P) + C_1 = kt + C_2$$

$$\ln(P) = kt + C$$

$$P(t) = e^{kt+C} = Ae^{kt}$$

Newton's Method of Series

$$\frac{dP}{dt} = k \cdot P$$

Now, assume that we can express *P* as a function of *t* by:

$$P(t) = \sum_{n=0}^{\infty} a_n t^n$$

Then $\dfrac{dP}{dt} = \displaystyle\sum_{n=1}^{\infty} n a_n t^{n-1}$ and our first equation gives

Newton's Method of Series

$$\frac{dP}{dt} = k \cdot P$$

$$\sum_{n=1}^{\infty} n a_n t^{n-1} = k \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} k a_n t^n$$

$$a_1 t^0 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + \cdots = ka_0 t^0 + ka_1 t + ka_2 t^2 + ka_3 t^3 + \cdots$$

Setting corresponding powers of *t* equal, gives the following

Newton's Method of Series

$$a_1 = ka_0$$

$$2a_2 = ka_1 \Rightarrow a_2 = \frac{k}{2}a_1 = \frac{k^2}{2}a_0$$

$$3a_3 = ka_2 \Rightarrow a_3 = \frac{k}{3}a_2 = \frac{k^3}{2\cdot 3}a_0$$

$$4a_4 = ka_3 \Rightarrow a_4 = \frac{k}{4}a_3 = \frac{k^4}{2\cdot 3\cdot 4}a_0$$

$$\vdots$$

$$na_n = ka_{n-1} \Rightarrow a_n = \frac{k}{n}a_{n-1} = \frac{k^n}{n!}a_0$$

Newton's Method of Series

$$P(t) = \sum_{n=0}^{\infty} a_n t^n = \sum_{n=0}^{\infty} a_0 \frac{k^n}{n!} t^n = a_0 \sum_{n=0}^{\infty} \frac{(kt)^n}{n!}$$

We will show later that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore,

$$P(t) = a_0 \sum_{n=0}^{\infty} \frac{(kt)^n}{n!} = a_0 e^{kt}$$

Newton's Method of Series

Example 2: Find a solution to the following equation

$$y'' + y = 0$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

Newton's Method of Series

$$y'' + y = 0$$

$$y'' = -y$$

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = -\sum_{n=0}^{\infty} a_n x^n$$

$$2a_2 + 3\cdot 2a_3 x + 4\cdot 3a_4 x^2 + 5\cdot 4a_4 x^3 + \cdots = -a_0 - a_1 x - a_2 x^2 - a_3 x^3 - \cdots$$

This gives us that

$$2a_2 = -a_0 \quad 4\cdot 3a_4 = -a_2 \quad 6\cdot 5a_6 = -a_4 \quad 8\cdot 7a_8 = -a_6$$

$$3\cdot 2a_3 = -a_1 \quad 5\cdot 4a_5 = -a_3 \quad 7\cdot 6a_7 = -a_5 \quad 9\cdot 8a_9 = -a_7$$

Newton's Method of Series

$$2a_2 = -a_0 \Rightarrow a_2 = -\frac{1}{2}a_0 = -\frac{1}{2!}a_0$$

$$3\cdot 2a_3 = -a_1 \Rightarrow a_3 = -\frac{1}{2\cdot 3}a_1 = -\frac{1}{3!}a_1$$

$$4\cdot 3a_4 = -a_2 \Rightarrow a_4 = -\frac{1}{4\cdot 3}a_2 = \frac{1}{2\cdot 3\cdot 4}a_0 = \frac{1}{4!}a_0$$

$$5\cdot 4a_5 = -a_3 \Rightarrow a_5 = -\frac{1}{5\cdot 4}a_3 = \frac{1}{2\cdot 3\cdot 4\cdot 5}a_1 = \frac{1}{5!}a_1$$

$$6\cdot 5a_6 = -a_4 \Rightarrow a_6 = -\frac{1}{6\cdot 5}a_4 = -\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}a_0 = -\frac{1}{6!}a_0$$

$$7\cdot 6a_7 = -a_5 \Rightarrow a_7 = -\frac{1}{7\cdot 6}a_5 = -\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}a_1 = -\frac{1}{7!}a_1$$

Newton's Method of Series

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

$$= a_0 + a_1 x - \frac{1}{2!} a_0 x^2 - \frac{1}{3!} a_1 x^3 + \frac{1}{4!} a_0 x^4 + \frac{1}{5!} a_1 x^5 - \frac{1}{6!} a_0 x^6 - \frac{1}{7!} a_1 x^7 + \cdots$$

$$= a_0 \left(1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots \right) + a_1 \left(x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots \right)$$

Let

$$c(x) = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots$$

and

$$s(x) = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7 + \cdots$$

Newton's Method of Series

This gives us that

$$y = a_0 c(x) + a_1 s(x)$$

Note that:

$$\begin{aligned}
[c(x)]^2 + [s(x)]^2 &= \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)^2 + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right)^2 \\
&= 1 - \frac{2}{2!}x^2 + \frac{1}{2!2!}x^4 + \frac{2}{4!}x^4 - \frac{2}{2!4!}x^6 - \frac{2}{6!}x^6 + x^2 - \frac{2}{3!}x^4 + \frac{1}{3!3!}x^6 + \frac{2}{5!}x^6 + \cdots \\
&= 1 + (1-1)x^2 + \left(\frac{1}{2!2!} + \frac{2}{4!} - \frac{2}{3!}\right)x^4 + \left(-\frac{2}{2!4!} - \frac{2}{6!} + \frac{1}{3!3!} + \frac{2}{5!}\right)x^6 + \cdots \\
&= 1 + 0x^2 + \left(\frac{1}{4} + \frac{1}{12} - \frac{1}{3}\right)x^4 + \left(-\frac{1}{24} - \frac{1}{360} + \frac{1}{36} + \frac{1}{60}\right)x^6 + \cdots \\
&= 1 + 0x^2 + \left(\frac{4}{12} - \frac{4}{12}\right)x^4 + \left(-\frac{16}{360} + \frac{16}{360}\right)x^6 + \cdots \\
&= 1
\end{aligned}$$

Newton's Method of Series

Also note that:

$$c(0) = 1$$

$$s(0) = 0$$

Further note that:

$$\frac{d}{dx}\cos(x) = -\cos(x)$$

$$\frac{d}{dx}\sin(x) = -\sin(x)$$

We can show that:

$$\mathrm{c}(x) = \cos(x)$$

$$\mathrm{s}(x) = \sin(x)$$

Your Turn

Solve by Newton's method (*i.e.*, find the first 5 non-zero terms):

1) $$y' = y + \frac{1}{1-x}$$

2) $$y' = 1 + xy$$

Solutions

$$y' = y + \frac{1}{1-x}$$

$$y(x) = a_o + (1 + a_o)x + \frac{1}{2}a_o x^2 + \frac{1}{3}\left(1 + \frac{1}{2}a_o\right)x^3 + \frac{1}{6}\left(\frac{1}{4}a_o - 1\right)x^4 + \frac{1}{6}\left(1 + \frac{1}{20}a_o\right)x^5 + \cdots$$

$$y' = 1 + xy$$

$$y(x) = a_o + x + \frac{1}{2}a_o x^2 + \frac{1}{3}x^3 + \frac{1}{8}a_o x^4 + \frac{1}{15}x^5 + \cdots$$

Solving Algebraic Equations

How can we solve an equation of the following form?

$$y^2 + 3y - 4 - x^2 y + 2x = 0$$

For this technique we do the same: assume that y has a power series expansion in terms of x.

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Where does this lead us?

Solving Algebraic Equations

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)^2 + 3\left(\sum_{n=0}^{\infty} a_n x^n\right) - 4 - x^2\left(\sum_{n=0}^{\infty} a_n x^n\right) + 2x = 0$$

How do we cube out a series? This seems to be the only sticking point to proceeding with this process.

We will make a couple of definitions and assumptions.

Solving Algebraic Equations

$$y = \sum_{n=0}^{\infty} a_n x^n$$

Given the above series define the *tail series* to be

$$t_n = \sum_{k=n}^{\infty} a_n x^n = a_n x^n + a_{n+1} x^{n+1} + a_{n+2} x^{n+2} + \cdots$$

We then have that

$$t_0 = y \quad and \quad t_n = a_n x^n + t_{n+1}$$

Solving Algebraic Equations

Given an algebraic equation we will write it as

$$F(x,y)=0$$

And we will begin our procedure by setting

$$F_0(x,y)=F(x,y)$$

The *k*th iteration of our process consists of three steps

1) Extract the x^k level equation from $F_k(x,t_k) = 0$. That is find the coefficients of x^k and set them equal to 0.
2) Solve the x^k level equation for a_k.
3) Substitute $t_k = a_k x^k + t_{k+1}$ into $F_k(x,t_k) = 0$ and use the Binomial Theorem to obtain the new equation $F_{k+1}(x,t_{k+1})$

Solving Algebraic Equations

Example:

$$F(x,y) = y^2 + 3y - 4 - x^2 y + 2x = 0$$

$$F_0(x,t_0): \qquad t_0^{\,2} + 3t_0 - 4 - x^2 t_0 + 2x = 0$$

Step 1: Since $t_0 = a_0 + a_1 x + a_2 x^2 + \ldots$ and the x^0 level equation ignores all non-constant terms in $F_0(x,t_0)$:

x^0 level:

$$a_0^{\,2} + 3a_0 - 4 = 0 \;\Rightarrow\; (a_0 + 4)(a_0 - 1) = 0 \;\Rightarrow\; a_0 = -4 \text{ or } a_0 = 1$$

This will give us two solutions and two equations

Solving Algebraic Equations

Solution 1: $a_0 = 1$

Now, replace t_0 with $1+t_1$ to get:

$$(1+t_1)^2 + 3(1+t_1) - 4 - x^2(1+t_1) + 2x = 0$$

$$1 + 2t_1 + t_1^2 + 3 + 3t_1 - 4 - x^2 - t_1x^2 + 2x = 0$$

$$5t_1 + t_1^2 - x^2 - t_1x^2 + 2x = 0$$

Ignore all terms with degree > 1.

$$5a_1 + 2 = 0$$

$$a_1 = -\frac{2}{5}$$

Solving Algebraic Equations

$$F_1(x,t_1) = 5t_1 + t_1{}^2 - x^2 - t_1 x^2 + 2x = 0$$

Now, replace t_1 with $-\frac{2}{5}x + t_2$

$$5t_1 + t_1{}^2 - x^2 - t_1 x^2 + 2x = 0$$

$$5\left(-\frac{2}{5}x + t_2\right) + \left(-\frac{2}{5}x + t_2\right)^2 - x^2 - \left(-\frac{2}{5}x + t_2\right)x^2 + 2x = 0$$

$$5t_2 + t_2{}^2 - \frac{4}{5}xt_2 - x^2 t_2 - \frac{21}{25}x^2 + \frac{2}{5}x^3 = 0$$

Ignore all terms with degree > 2.

$$5a_2 - \frac{21}{25} = 0$$

$$a_2 = \frac{21}{125}$$

Solving Algebraic Equations

$$F_2(x,t_2) = 5t_2 + t_2{}^2 - \frac{4}{5}xt_2 - x^2t_2 - \frac{21}{25}x^2 + \frac{2}{5}x^3 = 0$$

Now, replace t_2 with $\frac{21}{125}x^2 + t_3$

$$5t_3 + t_3{}^2 - \frac{4}{5}xt_3 - \frac{83}{125}x^2t_3 + \frac{166}{625}x^3 - \frac{2184}{15625}x^4 = 0$$

Ignore all terms with degree > 3.

$$5a_3 + \frac{166}{625} = 0$$

$$a_3 = -\frac{166}{3125}$$

Solving Algebraic Equations

So, we find a series expansion for y of the form:

$$y = \sum_{n=0}^{\infty} a_n x^n = 1 - \frac{2}{5}x + \frac{21}{125}x^2 - \frac{166}{3125}x^3 + \cdots$$

Your Turn

1) Find the solution corresponding to $a_0 = -4$

2) Find the first four terms of the infinite series expansion of y if $y^3 + xy = 1$.

Your Turn

1) We can solve the original equation for y and then find the series expansion for that. We should get the same answer.

$$y^2 + 3y - 4 - x^2 y + 2x = 0$$

$$y^2 + (3 - x^2)y + (2x - 4) = 0$$

$$y = \frac{(x^2 - 3) \pm \sqrt{(3 - x^2)^2 - 4(2x - 4)}}{2}$$

$$y = \frac{(x^2 - 3) \pm \sqrt{x^4 - 6x^2 - 8x + 25}}{2}$$

Your Turn

$$y_1 = \frac{x^2 - 3 + \sqrt{x^4 - 6x^2 - 8x + 25}}{2}$$

$$= 1 - \frac{2}{5}x + \frac{21}{125}x^2 - \frac{166}{3125}x^3 + \frac{304}{15625}x^4 + \cdots$$

$$y_2 = \frac{x^2 - 3 - \sqrt{x^4 - 6x^2 - 8x + 25}}{2}$$

$$= -4 + \frac{2}{5}x + \frac{104}{125}x^2 + \frac{166}{3125}x^3 - \frac{304}{15625}x^4 + \cdots$$

This is what we should have found with our other process.

Your Turn

2) Find the first four terms of the infinite series expansion of y if $y^3 + xy = 1$.

$$y^3 + xy - 1 = 0$$

$$y = \frac{1}{6}\sqrt[3]{108 + 12\sqrt{12x^3 + 81}} - \frac{2x}{\sqrt[3]{108 + 12\sqrt{12x^3 + 81}}}$$

$$y = 1 - \frac{1}{3}x + \frac{1}{81}x^3 + \frac{1}{243}x^4 + \cdots$$

Problem 2

$$F(x,y) = y^3 + xy - 1 = 0$$

Stage 0: $F_0(x,t_0) = t_0{}^3 + xt_0 - 1 = 0$

$$a_0{}^3 - 1 = 0$$

$$a_0 = 1$$

Stage 1: In $F_0(x,t_0)$ replace $t_0 = 1 + t_1$

$$F_1(x,t_1) = (1+t_1)^3 + x(1+t_1) - 1 = 0$$

$$F_1(x,t_1) = 3t_1 + 3t_1{}^2 + t_1{}^3 + xt_1 + x = 0$$

$$3a_1 + 1 = 0$$

$$a_1 = -\frac{1}{3}$$

Problem 2

Stage 2: In $F_1(x,t_1)$ replace $t_1 = -\frac{1}{3}x + t_2$

$$F_1(x,t_1) = 3t_1 + 3t_1^{\ 2} + t_1^{\ 3} + xt_1 + x = 0$$

$$F_2(x,t_2) = 3(-\frac{1}{3}x + t_2) + 3\left(-\frac{1}{3}x + t_2\right)^2 + \left(-\frac{1}{3}x + t_2\right)^3 + x\left(-\frac{1}{3}x + t_2\right) + x = 0$$

$$t_0 = 1 + t_1$$

$$= 3t_2 + 3t_2^{\ 2} + t_2^{\ 3} - xt_2 + \frac{1}{3}x^2 t_2 - xt_2^{\ 2} - \frac{1}{27}x^3 = 0$$

$$3a_2 = 0$$

$$a_2 = 0$$

Problem 2

Stage 3: In $F_2(x,t_2)$ replace $t_2 = 0x^2 + t_3 = t_3$

$$F_2(x,t_2) = 3t_2 + 3t_2^{\,2} + t_2^{\,3} - xt_2 + \frac{1}{3}x^2t_2 - xt_2^{\,2} - \frac{1}{27}x^3 = 0$$

$$F_3(x,t_3) = 3t_3 + 3t_2^{\,2} + t_2^{\,3} - xt_3 + \frac{1}{3}x^2t_3 - xt_3^{\,2} - \frac{1}{27}x^3 = 0$$

$$3a_3 - \frac{1}{27} = 0$$

$$a_3 = \frac{1}{81}$$

Problem 2

Stage 4: In $F_3(x,t_3)$ replace $t_3 = \frac{1}{81}x^3 + t_4$

$$F_3(x,t_3) = 3t_3 + 3t_2^{\ 2} + t_2^{\ 3} - xt_3 + \frac{1}{3}x^2 t_3 - xt_3^{\ 2} - \frac{1}{27}x^3 = 0$$

$$F_4(x,t_4) = 3\left(\frac{x^3}{81} + t_4\right) + 3\left(\frac{x^3}{81} + t_4\right)^2 + \left(\frac{x^3}{81} + t_4\right)^3 - x\left(\frac{x^3}{81} + t_4\right) +$$

$$= \frac{1}{3}x^2\left(\frac{x^3}{81} + t_4\right) - x\left(\frac{x^3}{81} + t_4\right)^2 - \frac{1}{27}x^3 = 0$$

$$= 3t_4 + 3t_4^{\ 2} + t_4^{\ 3} - xt_4 + \frac{x^2 t_4}{3} + \frac{2x^3 t_4}{27} - \frac{2x^4 t_4}{81} + \frac{x^6 t_4}{2187} - xt_4^{\ 2} +$$

$$= \frac{x^3 t_4^{\ 2}}{27} - \frac{x^4}{81} + \frac{x^5}{243} + \frac{x^6}{2187} - \frac{x^7}{6561} + \frac{x^9}{531441} = 0$$

$$3a_4 - \frac{1}{81} = 0$$

$$a_4 = \frac{1}{243}$$