# MATH 6101
# Fall 2008

## The Real Number System

# The Natural Numbers, ℕ

How did the concept of number arise?

Pythagoreans: “All is number”

- *Separated number from magnitude – numbers were considered more as distinct points*
- *Allows addition, subtraction, multiplication – what about division?*
- *Used “submultiple”: given the number 15 we say that 5 is a submultiple of 15 because* 3 x 5 = 15.

# Natural Numbers

Definition 1: *The submultiple, which is by its nature the smaller, is the number which when compared with the greater can measure it more times than one so as to fill it out exactly*.

*Magnitudes* are distinct from numbers but have different properties.

All numbers by definition were positive integer multiples of a base unit *but* ratios of lengths were shown not to have the property of being ratios of number.

# Rational Numbers

To the Greek mathematicians, the number system was augmented by those numbers/lengths which were the ratios of other lengths.

There were those that were the ratios of integers and those that were not commensurable with the unit.

The existence of numbers that were not rational was well known. (Plato)

# Euclid's Rational Numbers

Euclid defined the ratio of magnitudes.

When are two magnitudes are in the same ratio as a second pair of magnitudes?

Euclid – $a:b = c:d$ if given any natural numbers $n$ and $m$ we have

$na > mb$ if and only *if* $nc > md$
$na = mb$ if and only if $nc = md$
$na < mb$ if and only if $nc < md$

# Euclid's Rational Numbers

Euclid - Book VII Number

First he defines a unit, then a number is defined as being composed of a multitude of units, and parts and multiples are defined as for magnitudes.

Euclid, as earlier Greek mathematicians, did not consider 1 as a number. It was a unit and the numbers 2, 3, 4, … were composed of units.

# Euclid's Rational Numbers

Proportion for numbers

He shows that for numbers *a*, *b*, *c*, *d* that $a{:}b = c{:}d$ when the least numbers with ratio $a{:}b$ are the same as the least numbers with ratio $c{:}d$.

($^a/_b = {}^c/_d$ if they become the same when reduced to lowest terms)

An important result in Book VII is the Euclidean algorithm.

Note that Euclid never identified the ratio 2:1 with the number 2. These were two quite different concepts.

# Euclid's Numbers

Euclid left us with magnitudes which had lengths which, in modern terms, could be formed from positive integers by

- addition,
- subtraction,
- multiplication,
- division and
- taking square roots.

# Next Steps

- Arabic mathematicians (Omar Khayyam) showed how to solve all cubic equations by geometric methods.
- Fibonacci solved a cubic equation showing that its root was not formed from rationals and square roots of rationals as Euclid's magnitudes were.
- By the end of the 15th century mathematicians were using expressions built from positive integers by addition, subtraction, multiplication, division and taking $n$th roots. These are called *radical expressions*.

# Next Steps

By the 16th century rational numbers and roots of numbers were accepted as numbers.

Still a sharp distinction between these different types of numbers

Stifel (*Arithmetica Integra*, 1544) argues that irrationals must be considered valid numbers.

# Decimals

Stevin (*La Theinde*, 1585) introduced decimal fractions.

- Only finite decimals were allowed
- Only certain rationals to be represented exactly
- Other rationals could be represented approximately
- Intent was to calculate with approximate rational values.
- Notation taken up by Clavius and Napier

Others saw it as a backwards step - it could not even represent $^1/_3$ exactly.

# Decimals

- Stevin (*L'Arithmetique* , 1585)made a number of important advances in the study of the real numbers.
- He argued that all numbers such as square roots, irrational numbers, surds, negative numbers *etc* should all be treated as numbers and not distinguished as being different in nature.
- He argued against the Greek idea that 1 is not a number but a unit and the numbers 2, 3, 4, ... were composed of units.

# Another Century Passes

John Wallis (*A treatise of Algebra,* 1684) accepts the use of Stevin's decimals.

- Still only considers finite decimal expansions
- Realizes that with these one can approximate numbers (for him constructed from positive integers by addition, subtraction, multiplication, division, taking *n*th roots) as closely as one wishes.
- He understood there were proportions which did not fall within this definition of number, such as those associated with the area and circumference of a circle.

# Another Century Passes

- Wallis considered approximations by continued fractions and approximations by taking successive square roots.
  - This led to the study of infinite series but without the machinery to prove convergence
  - Real numbers became very much associated with magnitudes.
- No definition was really thought necessary
- Euler (*Complete introduction to algebra*, 1771) wrote -
  - *Mathematics, in general, is the science of quantity; or, the science which investigates the means of measuring quantity*.

# Euler’s Impact

- Euler defined the notion of quantity as that which can be continuously increased or diminished
  - Thought of length, area, volume, mass, velocity, time, *etc*. to be different examples of quantity
  - All could be measured by real numbers
- This led to a more abstract idea of quantity, a variable $x$ which need not necessarily take real values
- Symbolic mathematics took the notion of quantity too far
- By the beginning of the 19th century a more rigorous approach to mathematics by Cauchy and Bolzano began to provide the machinery to put the real numbers on a firmer footing.

# Cauchy

Cauchy (*Cours d'analyse*, 1821) did not give a careful definition of the real numbers.

He does say that a real number is the limit of a sequence of rational numbers (but he is assuming here that the real numbers are known)

This is not considered to be a definition of a real number, rather it is simply a statement of what he considers an *obvious* property.

(Avoids the question of how one is to define convergence of a sequence without assuming the existence of its limit)

# Bolzano

Bolzano (1817) showed that any bounded Cauchy sequence of real numbers had a least upper bound.

Later worked out his own theory of real numbers which he did not publish

His definition of a real number was made in terms of convergent sequences of rational numbers

Since these were unpublished they had little influence in the development of the theory of the real numbers.

# Transcendental Numbers

Up to this time there was no proof that numbers existed that were not the roots of polynomial equations with rational coefficients.

Clearly $\sqrt{2}$ is the root of a polynomial equation with rational coefficients, namely $x^2 = 2$

All roots of rational numbers arise as solutions of such equations.

# Transcendental Numbers

A number is called *transcendental* if it is not the root of a polynomial equation with rational coefficients.

The word *transcendental* is used as such a number transcends the usual operations of arithmetic

Mathematicians guessed for a long time that $\pi$ and $e$ were transcendental, but not proven up to the middle of the $19^{th}$ century.

# Liouville

Liouville's interest in transcendental numbers stemmed from reading a correspondence between Goldbach and Daniel Bernoulli.

Liouville wanted to prove that $e$ is transcendental but he did not succeed.

His contributions led him to prove the existence of a transcendental number in 1844 when he constructed an infinite class of such numbers using continued fractions.

# Liouville

In 1851 he published results on transcendental numbers removing the dependence on continued fractions.

The Liouville number

$$0.110001000000000000000001000\ldots$$

where there is a 1 in place $n!$ and 0 elsewhere is a transcendental number

# Liouville

1873 – Hermite showed *e* is transcendental

1882 – Lindemann showed π is transcendental

1900 – Hilbert proposed the following problem: *If* $\alpha, \beta$ *are algebraic and* $\alpha \neq 0, 1$ *and* $\beta$ *is irrational, prove that* $\alpha^{\beta}$ *is transcendental.*

1934 – Gelfond and Schneider proved the statement true.

We still do not know if $e \pm \pi$, $e^e$, $\pi^\pi$, $\pi e$ are transcendental.

We do know that at least one of $e+\pi$ and $\pi e$ is transcendental

# Known Transcendentals

We do know that the following are transcendental:

- $\ln 2$
- $\ln 3/\ln 2$
- $\sin(a)$, $\cos(a)$, $\tan(a)$, $a$ a nonzero rational
- $\arctan(x)/\pi$ for $x$ rational

$e^{\pi} = (e^{i\pi})^{-i} = (-1)^{i}$ is transcendental

$i^{i} = e^{-\pi/2}$ is transcendental

$2^{\sqrt{2}}$ is transcendental

# Foundations of Real Numbers

- Dedekind worked out his theory of Dedekind cuts in 1858 but it remained unpublished until 1872.
- Weierstrass gave his own theory of real numbers in his Berlin lectures beginning in 1865 but this work was not published.
- The first published contribution came in 1867 from Hankel a student of Weierstrass.

# Foundations of Real Numbers

Hankel suggested a complete change in our point of view regarding the concept of a real number:-

*Today number is no longer an object, a substance which exists outside the thinking subject and the objects giving rise to this substance, an independent principle, as it was for instance for the Pythagoreans. Therefore, the question of the existence of numbers can only refer to the thinking subject or to those objects of thought whose relations are represented by numbers. Strictly speaking, only that which is logically impossible (i.e. which contradicts itself) counts as impossible for the mathematician.*

# Foundations of Real Numbers

- Méray (*Remarques sur la nature des quantités*, 1869) considered Cauchy sequences of rational numbers which, if they did not converge to a rational limit, had what he called a *fictitious limit*. Considered the real numbers = the rational numbers and his fictitious limits.
- Heine (*Elemente der Functionenlehre*, 1872) published a similar notion though done independently of Méray. It was similar in nature with the ideas which Weierstrass had discussed in his lectures.
- Heine's system is one of the two standard ways of defining the real numbers today.

# Foundations of Real Numbers

- Heine looks at Cauchy sequences of rational numbers. He defines an equivalence relation on such sequences by defining $a_1$ , $a_2$ , $a_3$ , $a_4$ , ... and $b_1$, $b_2$ , $b_3$ , $b_4$ , ... to be equivalent if the sequence of rational numbers $a_1 - b_1$, $a_2 - b_2$ , $a_3 - b_3$ , $a_4 - b_4$ , ... converges to 0.
- He introduced arithmetic operations on his sequences and an order relation. (Care is needed to handle division since sequences with a non-zero limit might still have terms equal to 0.)

# Foundations of Real Numbers

- Cantor also published his version of the real numbers in 1872
- It followed a similar method to that of Heine.
- His numbers were Cauchy sequences of rational numbers and he used the term *determinate limit*.

# Foundations of Real Numbers

- Cantor realized that if he wanted the line to represent the real numbers then he has to introduce an axiom to recover the connection between the way the real numbers are now being defined and the old concept of measurement.
- He writes about a distance of a point from the origin on the line

  *… In order to complete the connection presented in this section of the domains of the quantities defined* [*his determinate limits*] *with the geometry of the straight line, one must add an axiom which simple says that every numerical quantity also has a determined point on the straight line whose coordinate is equal to that quantity, indeed, equal in the sense in which this is explained in this section.*

# Dedekind's Work

- When Dedekind realized that Heine and Cantor were about to publish their versions of a rigorous definition of the real numbers he decided that he should publish his ideas.
- In 1872 he published another definition of the real numbers.
- Dedekind considered all decompositions of the rational numbers into two sets $A_1$ , $A_2$ so that $a_1 < a_2$ for all $a_1$ in $A_1$ and $a_2$ in $A_2$. He called $(A_1, A_2)$ a *cut (Schnitt)*. If the rational $a$ is either the maximum element of $A_1$ or the minimum element of $A_2$ then he said the cut was produced by $a$.

# Dedekind's Work

- However not all cuts were produced by a rational.

  *In every case in which a cut* $(A_1, A_2)$ *is given that is not produced by a rational number, we create a new number, an irrational number a, which we consider to be completely defined by this cut; we will say that the number a corresponds to this cut or that it produces the cut.*
- He defined the usual arithmetic operations and ordering and showed that the usual laws apply.

# Further Work

- Another definition appeared in a book by Thomae in 1880. Thomae was a colleague of Heine and Cantor.
- He claimed that the real numbers defined in this way had a right to exist because … *the rules of combination abstracted from calculations with integers may be applied to them without contradiction.*

# Further Work

- Frege attacked these ideas of Thomae .
- He wanted to develop a theory of real numbers based on a purely logical base and attacked the philosophy behind the constructions which had been published.
- Thomae in 1898 "*The formal conception of numbers requires of itself more modest limitations than does the logical conception. It does not ask, what are and what shall the numbers be, but it asks, what does one require of numbers in arithmetic.*"

# Further Work

- Frege was still unhappy with the constructions of Weierstrass, Heine, Cantor, Thomae and Dedekind.
- How does one know that these constructions led to systems which would not produced contradictions?
- Frege, however, never completed his own version of a logical framework. His hopes were shattered when he learnt of Russell's paradox.

# Hilbert's Work

- Hilbert in 1900 took a different approach to defining the real numbers.
- He defined the real numbers to be a system with eighteen axioms. Sixteen of these axioms define what today we call an ordered field, while the other two were the Archimedean axiom and the completeness axiom.

# Hilbert's Work

- The Archimedean axiom stated that given positive numbers $a$ and $b$ then it is possible to add $a$ to itself a finite number of times so that the sum exceed $b$.
- The completeness property says that one cannot extend the system and maintain the validity of all the other axioms.

# Hilbert’s Work

- This was totally new since all other methods built the real numbers from the known rational numbers.
- Hilbert's numbers were unconnected with any known system. It was impossible to say whether a given mathematical object was a real number.
- There was no proof that any such system actually existed

# Fields

A field is a set $F$ with two binary operations + and × for which the following hold. For any elements $a$, $b$, $c$ in $F$:

a. $a + b \in F, a \times b \in F$

b. $(a + b) + c = a + (b + c), (a \times b) \times c = a \times (b \times c)$

c. $a + b = b + a, a \times b = b \times a$

d. $a \times (b + c) = (a \times b) + (a \times c)$

e. There are distinct elements $0,1 \in F$ so that $a + 0 = a$, and $a \times 1 = a$

f. There is $-a \in F$ so that $a + (-a) = 0$

g. If $a \neq 0$, then there is $a^{-1} \in F$ so that $a \times a^{-1} = 1$.

# Field Consequences

Lemma: If $F$ is a field and $a, b, c \in F$, then:

*a.* $a + b = a + c$ implies that $a = b$

*b.* $a \times 0 = 0$

c. $(-a) \times b = -(a \times b)$

d. $(-a) \times (-b) = a \times b$

*e.* $a \times b = a \times c$ *and* $a \neq 0$ then $b = c$

f. If $a \times \mathrm{b} = 0$, then either $a = 0$ or $b = 0$.

# Ordered Field

A field $F$ is an *ordered field* if there is a binary relation $\leq$ on its elements that has the following properties for $a, b, c \in F$:

a. Either $a \leq b$ or $b \leq a$
b. If $a \leq b$ and $b \leq a$ then $a = b$
c. If $a \leq b$ and $b \leq c$ then $a \leq c$
d. If $a \leq b$ then $a + c \leq b + c$
e. If $a \leq b$ and $c \leq 0$ then $a \times c \leq b \times c$

# Bounded Sets

If $S$ is a finite set of elements in an ordered field, then max $S$ denotes the largest element in $S$.

If $S$ is a set of elements in an ordered field, and $b$ is any element so that $r \leq b$ for any $r$ in $S$ then $b$ is called an *upper bound* for $S$ and $S$ is said to be *bounded above* by $b$.

If $S$ is a set of elements in an ordered field, and $b$ is any element so that $b \leq r$ for any $r$ in $S$ then $b$ is called a *lower bound* for $S$ and $S$ is said to be *bounded below* by $b$.

If $S$ is a set of elements in an ordered field, and $b$ is any element so that $|r| \leq b$ for any $r$ in $S$ then $b$ is called a *bound* for $S$ and $S$ is said to be *bounded* by $b$.

<u>Lemma</u>: If $S_1, S_2, \ldots, S_n$ are bounded sets, then $S_1 \cup S_2 \cup \ldots \cup S_n$ is also bounded.

# Least Upper Bounds

If $b^*$ is an upper bound for a set $S$ so that $b^* \leq b$ for every upper bound $b$ of $S$ then $b^*$ is called a *least upper bound* for $S$.

Lemma: $b^*$ is unique.

If $d^*$ is a lower bound for a set $S$ so that $d \leq d^*$ for every lower bound $d$ of $S$ then $d^*$ is called the *greatest lower bound* for $S$.

***Question***: Is the empty set bounded above? Does the empty set have a least upper bound?

# Different Fields - Different Results

Let $S = \{ r \in \mathbb{R} \mid r^2 \leq 2\}$. $S$ is bounded . What is the least upper bound for $S$?

Now, let $T = \{ r \in \mathbb{Q} \mid r^2 \leq 2\}$. $T$ is bounded . What is the least upper bound for $T$ in $\mathbb{Q}$?

This is the essential difference between the reals and rationals. This is embodied in the following axiom satisfied by the reals.

***Axiom***: (Completeness or Least Upper Bound Axiom) Every non-empty set of real numbers that is bounded above has a least upper bound.

# The Real Numbers

Following Hilbert's lead we then define the real number system to be an ordered field that satisfies the Least Upper Bound Axiom.

***Proposition***: Given any real number $r$, there is an integer $n$ so that $n \leq r < n + 1$.

***Proposition***: Given $r$ is a non-zero rational number and $a$ is an irrational number then the numbers $a \pm r$, $ar$, $a/r$, and $r/a$ are all irrational.

***Proposition***: If $a < b$ are real numbers, then there exists both a rational and an irrational number between $a$ and $b$.