MATH 6101
Fall 2008

Functions, Sequences and Limits

The Topology of the Reals

We will make some simple definitions. Let a and b be any two real numbers with $a < b$.

$$(a,b) = \{ x \in R \mid a < x < b\}$$
$$[a,b] = \{ x \in R \mid a \le x \le b\}$$
$$(a,b] = \{ x \in R \mid a < x \le b\}$$
$$[a,b) = \{ x \in R \mid a \le x < b\}$$
$$(a,\infty) = \{ x \in R \mid a < x \}$$
$$[a,\infty) = \{ x \in R \mid a \le x \}$$
$$(-\infty,b) = \{ x \in R \mid x < b \}$$
$$(-\infty,b] = \{ x \in R \mid x \le b \}$$

Topology of the Reals

If $r \in R$ then a *neighborhood* of r is ***<u>an</u>*** open interval (a,b) so that $r \in (a,b)$.

The neighborhood is *centered* at r if

$$r = (a + b)/2$$

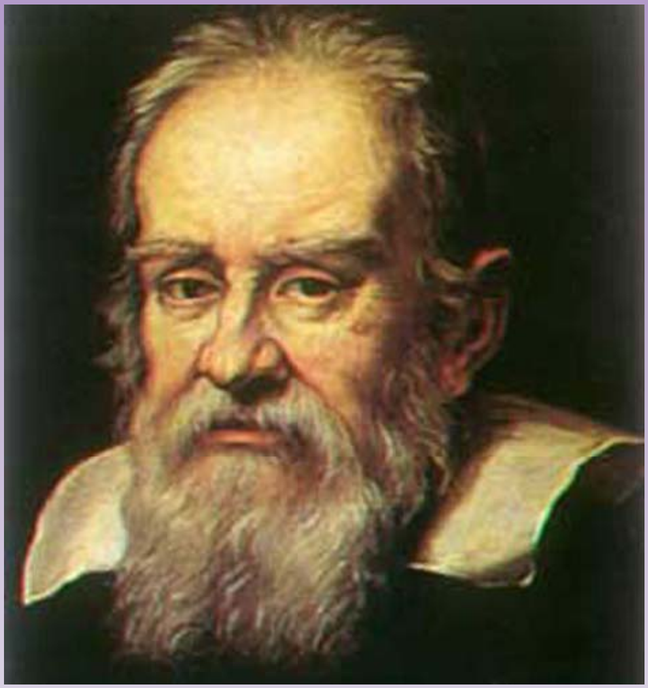
If ε and a are reals, then the *ε-neighborhood* of a is the interval $(a - \varepsilon, a + \varepsilon)$

Functions

Nicole Oresme – 1350 – described the laws of nature as laws giving a dependence of one quantity on another.

History of Function

Galileo – 1638 – studies of motion contain the clear understanding of a relation between variables



History of Function

Descartes - an equation in two variables, geometrically represented by a curve, indicates a dependence between variable quantities

Euclid’s Rational Numbers

Newton – showed how functions arise from infinite power series

Leibniz – 1673 – the first to use the term *function*. He took function to designate, in very general terms, the dependence of geometrical quantities on the shape of a curve.

History of Function

- Jean Bernoulli - 1718 - function of a variable as a quantity that is composed in some way from that variable and constants
- Euler – 1748 - A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.
- Euler – 1755 - If some quantities so depend on other quantities that if the latter are changed the former undergoes change, then the former quantities are called functions of the latter.

History of Function

- Cauchy – 1821 – still thinking of a function in terms of a formula (either explicit or implicit)
- Fourier – 1822 – introduced general Fourier series but fell back on old definitions
- Dirichlet – 1837 – defined general function and continuity (in modern terms)
- Weierstrauss – 1885 – any continuous function is the limit of a uniformly convergent sequence of polynomials
- Goursat – 1923 – modern definition

Definitions

Bernoulli – 1718 – *One calls here a function of a variable a quantity composed in any manner whatever of this variable and constants*.

Basically this meant +, –, ×, ÷, √, logs and sines.

They would say that $f(x)$ depended *analytically* on the variable x.

Definitions

Fourier – 1822 – *In general the function f(x) represents a succession or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there area an equal number of ordinates f(x). All have actual numerical values, either positive or negative or null. We do not suppose these ordinates to be subject to common law; the succeed each other in any manner whatever, and each of them is given as it were a single quantity.*

Definitions

Fourier removed the requirement of “analytic” from the definition. It was not widely accepted for years.

Definitions

Dirichlet – 1837 – *Let us suppose that a and b are two definite values and x is a variable quantity which is to assume, gradually, all values located between a and b. Now, if to each x there corresponds a unique, finite y …, then y is called a … function of x for this interval. It is, moreover, not at all necessary, that y depends on x in this whole interval according to the same law; indeed, it is not necessary to think of only relations that can be expressed by mathematical operations.*

Definitions

Every "Bernoulli" function is a "Fourier" or a "Dirichlet" function.

Dirichlet:

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational and } 0 \le x \le 1 \\ 0 & \text{if } x \text{ is irrational and } 0 \le x \le 1 \end{cases}$$

Another "Bad Example"

d'Alembert was working on the problem of describing a vibrating string. The initial position for the string is not the graph of any analytical expression.

A More Modern Definition

Let D be a set of real numbers. A function

$$f: D \rightarrow R$$

is a rule that assigns a number $f(x)$ to every element x of D.

Modern Set Theory Definition

A function *f* is an ordered triple of sets (F,X,Y) with restrictions, where F (the **graph**) is a set of ordered pairs (x,y), X (the **source**) contains all the first elements of F and perhaps more, and Y (the **target**) contains all the second elements of F and perhaps more.

The most common restrictions are that F pairs each x with just <u>*one*</u> y, and that X is just the set of first elements of F and no more.

When *no* restrictions are placed on F, we speak of a *relation* between X and Y rather than a function. The relation is "single-valued" when the first restriction holds: $(x,y_1) \in F$ and $(x,y_2) \in F$ together imply $y_1 = y_2$.

Relations that are not single valued are sometimes called *multivalued* functions. A relation is *total* when a second restriction holds: if $x \in X$ then $(x,y) \in F$ for some y. Thus we can also say that

A function from X to Y is a single-valued, total relation between X and Y.

Sequences

Let N = the set of natural numbers (it will not matter if it starts with 0 or with 1).

A sequence is a function a: $N \rightarrow$ R.

We will normally denote a sequence by its set of outputs $\{a_n\}$, where $a_n = a(n)$.

Occasionally you will see a_0, a_1, a_2, a_3, … or $\{a_n\}^{\infty}{}_{n=0}$

Examples

1) $\{1,2,3,4,5,6,\ldots\}$ – an arithmetic progression $(f(n) = n)$
2) $\{a + bn \mid n=0,1,2,3,\ldots\}$ – a different type of arithmetic progression – $(f(n) = a + bn)$
3) $\{a^0,a^1,a^2,a^3,a^4,\ldots\}$ – a geometric progression $(f(n) = a^n)$
4) $\{1, 1/2, 1/3, 1/4, 1/5,\ldots\}$ - $(f(n) = 1/n)$
5) $f(n) = a_n = (-1)^n$. Note that the range is $\{-1,1\}$

Examples

1) $f(n) = a_n = \cos(\pi n/3)$
$a_1 = \cos(\pi/3) = \cos 60° = ½$
$\{a_n\}$ = {½, -½, -1, -½, ½, 1,½, -½, -1, -½, ½, 1, …}. The function takes on only a finite number of values, but the sequence has an infinite number of elements.

2) $f(n) = a_n = n^{1/n}$,
$\{1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \ldots\}$ = {1, 1.41421, 1.44225, 1.41421, 1.37973, 1.34801, 1.32047, 1.29684, 1.27652, 1.25893,…}
Also a_{100}=1.04713, $a_{10,000}$ = 1.00092

3) $b_n = (1+1/n)^n$
$\{2, (3/2)^2, (4/3)^3, (54)^4, \ldots\}$ = {2, 2.25, 2.37037, 2.44141, 2.48832, 2.52163, 2.54650, 2.56578, 2.58117, 2.59374,…}
Also a_{100}=2.74081 and $a_{10,000}$ = 2.71815

Almost all …

Definition: It is said that *almost all the terms* of the sequence $\{a_n\}$ have a certain property provided that there is an index N such that $\{a_n\}$ possesses this property whenever $n \geq N$.

Convergence

Definition 1: A sequence of real numbers is said to *converge* to a real number L if for every $\varepsilon > 0$ there is an integer $N > 0$ such that if $k > N$ then $|a_k - L| < \varepsilon$.

Definition 2: A sequence of real numbers is said to *converge* to a real number L if every neighborhood of L contains almost all of the terms of $\{a_n\}$.

The number L is called the *limit* of the sequence.

Convergence

Lemma 1: The sequence $\{1/n\}$ converges to 0.

Proof : Let (a,b) be any neighborhood of 0. This means that $a < 0 < b$. Let $N>[1/b]$, be an integer greater than $1/b$. Then $1/N < b$ and for every integer $n > N$, we have that

$$a < 0 < 1/n < 1/N < b$$

and (a,b) contains almost all of the elements of the sequence. Thus, the sequence converges to 0.

Convergence

Lemma 1: The sequence $\{1/n\}$ converges to 0.

Proof : You prove this using Definition 1.

Convergence

Definition: A sequence is *convergent* if it has a limit. If it is not convergent it is called *divergent*.

Lemma 2: The sequence $\{a_n\}$ converges to L if and only if every neighborhood of L that is centered at L contains almost all of the terms of the sequence.

Note that this tells us that the two definitions are the same.

Example

Let $a_n = n/2^n$. $\{a_n\} = \{1/2, 2/2^2, 3/2^3, 4/2^4, \ldots\}$
Educated guess: $\{a_n\}$ -> 0.

Let ε = 0.1, 0.01, 0.001, 0.0001, 0.00001.

We need to find an integer N so that

$$| N/2^N - 0 | < \varepsilon$$

Look in the table of values. Note that for $N = 6$ the above is true if $\varepsilon = 0.1$

ε	**N**
1	N>0
0.1	N>5
0.01	N>9
0.001	N>14
0.0001	N>18
0.00001	N>22

Theorem (Convergent sequences are bounded)

Let $\{a_n\}$ be a convergent sequence. Then the sequence is bounded, and the limit is unique.

Proof:

(i)Uniqueness: Suppose the sequence has two limits, L and K. Let $\varepsilon > 0$. There is an integer N_K such that $|a_n - K| < \varepsilon/2$ if $n > N_K$.

Also, there is an integer N_L such that $|a_n - L| < \varepsilon/2$ if $n > N_L$.

By Triangle Inequality:

$$|L - K| < |a_n - L| + |a_n - K| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

if $n >\max\{N_K, N_L\}$.

Therefore $|L - K| < \varepsilon$ for any $\varepsilon > 0$. But this means that $L = K$.

Theorem(Convergent sequences are bounded)

Proof:

(ii) Boundedness. Since the sequence converges, choose any $\varepsilon > 0$. Specifically take $\varepsilon = 1$. There is N so that

$$| a_n - L | < 1 \text{ if } n > N.$$

Fix N. Then

$$| a_n | \leq | a_n - L | + |L| < 1 + |L| = P \text{ for all } n > N.$$

Let $M=\max\{|a_1|,|a_2|,\ldots|a_N|, P\}$. Thus $|a_n| < M$ for all n, which makes the sequence bounded.

Theorem: *If* $\{a_n\} \longrightarrow L$, $\{b_n\} \longrightarrow M$ *and* α *is a real number, then*

1. $\lim_{n\longrightarrow\infty} \alpha = \alpha$.
2. $\lim_{n\longrightarrow\infty} (a_n \pm b_n) = L \pm M$
3. $\lim_{n\longrightarrow\infty} (a_n \times b_n) = L \times M$
4. $\lim_{n\longrightarrow\infty} (\alpha a_n) = \alpha L$
5. *If* $a_n \leq b_n$ *for all* $n \geq m$, *then* $L \leq M$
6. *If* $b_n \neq 0$ *for all* n *and if* $M \neq 0$, *then* $glb\{|b_n|\} > 0$.
7. $\lim_{n\longrightarrow\infty} (a_n / b_n) = L/M$, *provided* $M \neq 0$.

Proof:

1. $lim_{n \longrightarrow \infty} \alpha = \alpha$

   Since $\alpha - \alpha = 0$, for any $\varepsilon > 0$, $| \alpha - \alpha | < \varepsilon$ and we are done.

2. $lim_{n \longrightarrow \infty} (a_n \pm b_n) = L \pm M$

   Do this for the sum. The difference is similar.

   Let $\varepsilon > 0$, there exist N_a and N_b so that

   $$| a_n - L | < \varepsilon/2 \quad \text{if } n > N_a \; and$$

   $$| b_n - M | < \varepsilon/2 \quad \text{if } n > N_b.$$

   Let $K = \max\{N_a, N_b\}$, then if $n > K$

   $$|(a_n+b_n) - (L+M)| \le |a_n - L|+|b_n - M| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

3. $lim_{n \longrightarrow \infty} (a_n \times b_n) = L \times M$

Note:

$$|(a_n b_n) - (LM)| \leq |(a_n - L)\, b_n + \mathrm{L}(b_n - M)|$$
$$\leq |(a_n - L)\, b_n| + |\mathrm{L}(b_n - M)|$$
$$= |(a_n - L)||b_n| + |\mathrm{L}||(b_n - M)|$$

Then use the fact that $\{b_n\}$ is bounded.

4. $lim_{n \longrightarrow \infty} (\alpha a_n) = \alpha L$

Consider ε/α if $\alpha \neq 0$. If $\alpha = 0$ this is easy.

5. *If* $a_n \le b_n$ *for all* $n \ge m$, *then* $L \le M$

6. *If* $b_n \neq 0$ *for all* n *and if* $M \neq 0$, *then* $glb\{|b_n|\} > 0$.

Let $\varepsilon = |M|/2 > 0$. $\{b_n\} \longrightarrow M$ so there is N so that if $n > N$ then $|b_n - M| < |M|/2$.

So if $n > N$ we must have $|b_n| \ge |M|/2$.

If not by the Triangle Inequality

$$|M| = |M - b_n + b_n| \le |M - b_n| + |b_n|$$
$$< |M|/2 + |M|/2 = |M|$$

So set

$$m = \min\{|M|/2, |b_1|, |b_2|, \ldots, |b_N|\}.$$

Then $m > 0$ and $|b_n| \ge m$ for all n

7. $lim_{n \to \infty} (a_n / b_n) = L/M$, *provided* $M \neq 0$.

Reduce to $lim_{n \to \infty} (1/b_n) = 1/M$ – How?

Let $\varepsilon > 0$. By (6) there is $m > 0$ so that $| b_n | \geq m$. Since $\{b_n\}$ is convergent there is N so that if $n > N$

$$|M - b_n| < \varepsilon\, m\, |M|$$

Then for $n > N$

$$| 1/b_n - 1/M| = | b_n - M|/|b_n M|$$

$$\leq | b_n - M|/(m|M|) < \varepsilon$$

Example

$$\lim_{n\to\infty}\frac{3n^4+4n^3-7n^2-5280n+3216547}{7n^4+5588741226n^2-7}$$

$$=\lim_{n\to\infty}\frac{3\frac{n^4}{n^4}+4\frac{n^3}{n^4}-7\frac{n^2}{n^4}-5280\frac{n}{n^4}+3216547\frac{1}{n^4}}{7\frac{n^4}{n^4}+5588741226\frac{n^2}{n^4}-7\frac{1}{n^4}}$$

$$=\lim_{n\to\infty}\frac{3+\frac{4}{n}-\frac{7}{n^2}-\frac{5280}{n^3}+\frac{3216547}{n^4}}{7+\frac{5588741226}{n^2}-\frac{7}{n^4}}=\frac{3}{7}$$

The Squeeze Theorem

<u>Theorem</u>: *If* $\{a_n\} \longrightarrow L$, $\{b_n\} \longrightarrow L$ *and*

$$a_n \le c_n \le b_n \text{ for all } n \ge m$$

Then $\{c_n\} \longrightarrow L$.

The Power Theorem

<u>Theorem</u>: *Let a be fixed. Then*

$$\lim_{n\to\infty} a^n = \begin{cases} 0 & \text{if } |a| < 1 \\ 1 & \text{if } a = 1 \\ dne & \text{if } |a| > 1 \\ dne & \text{if } a = -1 \end{cases}$$

Find

$$\lim_{n\to\infty}\frac{3n^2+5n+1}{n+1}$$

Find

$$\lim_{n\to\infty}\frac{3n^2+5n+1}{n^2+1}$$

Find

$$\lim_{n \to \infty} \frac{3n^2 + 5n + 1}{n^3 + 1}$$

Find

$$\lim_{n\to\infty} \frac{0.5^n + 3\sin(n)}{\sqrt{n}}$$

Find

$$\lim_{n\to\infty} \frac{2^n - 1}{3^n + 1}$$

Find

$$\lim_{n\to\infty} \frac{2^n + 1}{3^n - 1}$$

Find

$$\lim_{n\to\infty}\frac{3^n+2^n}{3^n-2^n}$$

Find

$$\lim_{n\to\infty} \frac{3^n + 4^{n-3}}{5^{n+2} - 2^{n+4}}$$

Find

$$\lim_{n\to\infty} \frac{4^{2n-3} + 2^{5n+6}}{5^{3n-2} - 3^{n+10}}$$

Find

$$\lim_{n \to \infty} n^n$$

Find

$$\lim_{n \to \infty} \frac{1}{n^n}$$

Find

$$\lim_{n\to\infty}\left(1-\left|\frac{\sin(n)}{n}\right|\right)$$

Find

$$\lim_{n \to \infty} \frac{n}{2^n}$$

Find

$$\lim_{n\to\infty}\frac{n^2}{2^n}$$

Find

$$\lim_{n\to\infty}\frac{n!}{n^n}$$

Find

$$\lim_{n\to\infty} \frac{n!}{2^n}$$