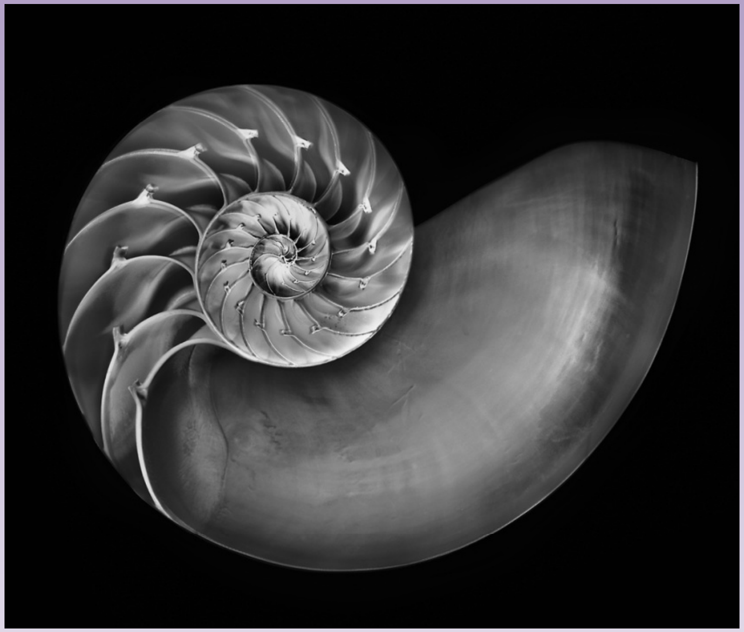
MATH 6101
Fall 2008

Series and a Famous Unsolved Problem



Problems

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)(2n+1)} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \frac{1}{7\cdot 9} + \cdots$$

Problems

$$\sum_{n=1}^{\infty} \frac{1}{n(n+4)}$$

$$\frac{25}{48}$$

Problems

$$\sum_{n=1}^{\infty} \frac{n}{(n+1)!}$$

1

Problems

$$2 + \frac{3}{2^3} + \frac{4}{3^3} + \frac{5}{4^3} + \cdots + \frac{n+1}{n^3} + \cdots$$

Problems

$$1+\frac{1}{2^2}+\frac{1}{3^3}+\frac{1}{4^4}+\cdots=\sum_{n=1}^{\infty}\frac{1}{n^n}$$

Problems

$$\sum_{n=1}^{\infty} \frac{n^n}{n!}$$

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Problems

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{(n!)^2}$$

Problems

$$\sum_{n=1}^{\infty} \frac{(2n!)}{n^4}$$

Problems

$$\sum_{n=0}^{\infty}\left(\frac{n}{n+1}\right)^{n}$$

Problems

$$\sum_{n=1}^{\infty} n r^n, \quad |r| < 1$$

Problems

Show $$\sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2}$$

Problems

Find $$\sum_{n=1}^{\infty} \frac{n}{2^n}$$

Prime Numbers

Statement:

There are an infinite number of prime numbers.

Proof A (Euclid):

Assume not, that is assume that there are a finite number of prime numbers: $p_1, p_2, p_3,\ldots,p_n$.

Let $M = (p_1 \times p_2 \times p_3 \times \ldots \times p_n) + 1$. Note that M is not divisible by $p_1, p_2, p_3,\ldots$ or p_n, since none of these divide 1. Thus, M is divisible only by itself and 1. Therefore, M is prime and not in the list – a contradiction.

Prime Numbers

Proof B (Euler): (Using series)

Assume that there are a finite number of prime numbers: 2, 3, 5,…,p, where p is the largest prime.

$$\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n = \frac{1}{1-1/2} = 2$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-1/3} = \frac{3}{2}$$

$$\vdots$$

$$\sum_{n=0}^{\infty}\left(\frac{1}{p}\right)^n = \frac{1}{1-1/p} = \frac{p}{p-1}$$

Prime Numbers

Thus the product is finite:

$$\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\right)\times\left(\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\right)\times\cdots\times\left(\sum_{n=0}^{\infty}\left(\frac{1}{p}\right)^n\right)=(2)\left(\frac{3}{2}\right)\cdots\left(\frac{p}{p-1}\right)$$

What does the product on the left hand side really look like?

$$\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\right)\times\left(\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\right)\times\cdots\times\left(\sum_{n=0}^{\infty}\left(\frac{1}{p}\right)^n\right)=$$

$$\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\right)\left(1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\cdots\right)\cdots\left(1+\frac{1}{p}+\frac{1}{p^2}+\frac{1}{p^3}+\cdots\right)$$

Prime Numbers

If we distribute the multiplication through the addition we get an infinite sum each of whose summands looks like

$$\frac{1}{2^{d_2}}\frac{1}{3^{d_3}}\frac{1}{5^{d_5}}\cdots\frac{1}{p^{d_p}}=\frac{1}{2^{d_2}\cdot 3^{d_3}\cdot 5^{d_5}\cdots p^{d_p}}$$

where each of the d_k is a non-negative integer.

Each summand occurs exactly once.

The denominator is a positive integer and each positive integer appears exactly once. Therefore

$$\left(\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\right)\times\left(\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n\right)\times\cdots\times\left(\sum_{n=0}^{\infty}\left(\frac{1}{p}\right)^n\right)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\cdots$$

Prime Numbers

The right hand side is the harmonic series which diverges and is not a real number. This is the needed contradiction.

Prime Numbers

This wasn't enough for Euler. He wanted to see what size the set of primes was. He showed the sum of the reciprocals of all prime numbers diverges.

Theorem: $\sum 1/p$ diverges.

Proof:

From above we know that

$$\sum_{n=1}^{\infty}\frac{1}{n}=\prod_{p\text{ prime}}\left(\frac{1}{1-p^{-1}}\right)$$

Prime Numbers

Take the log of both sides.

$$\ln\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)=\ln\left[\prod_{p\text{ prime}}\left(\frac{1}{1-p^{-1}}\right)\right]=\sum_{p\text{ prime}}\ln\left(\frac{1}{1-p^{-1}}\right)=\sum_{p\text{ prime}}-\ln\left(1-p^{-1}\right)$$

Euler now uses the Taylor series for the logarithm.

$$\ln(1-x)=-x-\frac{1}{2}x^2-\frac{1}{3}x^3-\frac{1}{4}x^4-\cdots-\frac{1}{n}x^n-\cdots$$

$$-\ln\left(1-p^{-1}\right)=p^{-1}+\frac{1}{2}p^{-2}+\frac{1}{3}p^{-3}+\frac{1}{4}p^{-4}+\cdots+\frac{1}{n}p^{-n}+\cdots$$

$$=\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\cdots+\frac{1}{np^n}+\cdots$$

Prime Numbers

Take the log of both sides.

$$\ln\left(\sum_{n=1}^{\infty}\frac{1}{n}\right) = \sum_{p\text{ prime}}\left(\frac{1}{p}+\frac{1}{2p^2}+\frac{1}{3p^3}+\frac{1}{4p^4}+\cdots+\frac{1}{np^n}+\cdots\right)$$

$$= \sum_{p\text{ prime}}\left(\frac{1}{p}\right)+\sum_{p\text{ prime}}\frac{1}{p^2}\left(\frac{1}{2}+\frac{1}{3p}+\frac{1}{4p^2}+\cdots+\frac{1}{np^{n-2}}+\cdots\right)$$

$$< \sum_{p\text{ prime}}\left(\frac{1}{p}\right)+\sum_{p\text{ prime}}\frac{1}{p^2}\left(1+\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^n}+\cdots\right)$$

$$= \sum_{p\text{ prime}}\left(\frac{1}{p}\right)+\sum_{p\text{ prime}}\frac{1}{p(p-1)}$$

$$= \sum_{p\text{ prime}}\left(\frac{1}{p}\right)+C$$

Prime Numbers

where C is a number < 1. Since $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, we know that

$$\ln\left(\sum_{n=1}^{\infty}\frac{1}{n}\right)$$

diverges. Thus, $\sum_{p\text{ prime}}\left(\frac{1}{p}\right)$ diverges.

We do know that if you sum the reciprocals of the twin primes that that sum is finite and slightly bigger than 1. (Apery's constant)

Probability

Proposition:

If two positive integers are chosen independently and randomly, then the probability that they are relatively prime is $6/\pi^2$.

Proof:

Let p be prime. Let n be randomly chosen integer.

$$\alpha = \text{Prob}(p|n) = 1/p.$$

Probability

If m and n are independently and randomly chosen

$$\text{Prob}(p|m \text{ and } p|n) = 1/p^2.$$

Thus,

$$\text{Prob}(p \text{ not divide both } m \text{ and } n) = 1 - 1/p^2$$

List the primes: $p_1, p_2, p_3, \ldots, p_k, \ldots$ and let

$P_k = \text{Prob}(p_k \text{ not divide both } m \text{ and } n) = 1 - 1/p_k^2$

Claim: Divisibility by p_i and p_j are independent.

Let P be the probability that m and n are relatively prime.

Probability

$$P = P_1 P_2 P_3 \cdots P_k \cdots$$

So

$$\frac{1}{P} = \frac{1}{P_1 P_2 \cdots P_k \cdots}$$

$$= \frac{1}{1-\frac{1}{2^2}} \cdot \frac{1}{1-\frac{1}{3^2}} \cdot \frac{1}{1-\frac{1}{5^2}} \cdot \cdots \cdot \frac{1}{1-\frac{1}{p_k^2}} \cdots$$

$$= \left(1 + \frac{1}{2^2} + \frac{1}{(2^2)^2} + \frac{1}{(2^2)^3} + \right) \times \left(1 + \frac{1}{3^2} + \frac{1}{(3^2)^2} + \frac{1}{(3^2)^3} + \right) \times$$

$$\left(1 + \frac{1}{5^2} + \frac{1}{(5^2)^2} + \frac{1}{(5^2)^3} + \right) \times \cdots$$

Probability

$$\frac{1}{P} = \left(1 + \frac{1}{2^2} + \frac{1}{(2^2)^2} + \frac{1}{(2^2)^3} + \right) \times \left(1 + \frac{1}{3^2} + \frac{1}{(3^2)^2} + \frac{1}{(3^2)^3} + \right) \times$$

$$\left(1 + \frac{1}{5^2} + \frac{1}{(5^2)^2} + \frac{1}{(5^2)^3} + \right) \times \cdots$$

$$= \left(1 + \frac{1}{2^2} + \frac{1}{(2^2)^2} + \frac{1}{(2^3)^2} + \right) \times \left(1 + \frac{1}{3^2} + \frac{1}{(3^2)^2} + \frac{1}{(3^3)^2} + \right) \times$$

$$\left(1 + \frac{1}{5^2} + \frac{1}{(5^2)^2} + \frac{1}{(5^3)^2} + \right) \times \cdots$$

$$= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots = \frac{\pi^2}{6}$$

Riemann Zeta Function

In 1859 Reimann defined a differentiable function of a complex variable $\zeta(s)$ by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots + \frac{1}{n^s} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Riemann knew the value of this function at certain values of *s*.

$\zeta(0)$ does not exist. (Why?)

$\zeta(1)$ does not exist. (Why?)

$\zeta(2) = \pi^2/6$

$\zeta(4) = \pi^4/90$

Riemann Zeta Function

Euler had computed $\zeta(2n)$ for $n = 1, 2, 3, \ldots, 13$.
We saw last time that

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{2^{2k-1}\left|B_{2k}\right|\pi^{2k}}{(2k)!}$$

Riemann showed that $\zeta(s)$ gives a lot of information about the distribution of primes.

Riemann Zeta Function

Question: Where is $\zeta(s) = 0$?

It can be shown that when it has been extended to all complex numbers, except $\text{Re}(s) = 1$, then it is trivially seen to be zero at the negative even integers.

Riemann proved that $\zeta(s) = 0$ when s falls inside the infinite strip bounded by the lines $x = 0$ and $x = 1$.

Prime Number Theorem

<u>Theorem</u>:

For every real number x let $\pi(x)$ *be the number of prime numbers less than x and let*

$$Li(x) = \int_2^x \frac{dt}{\ln(t)}$$

Then

$$\lim_{x \to \infty} \frac{\pi(x)}{Li(x)} = 1$$

This was proven by Hadamard and Poussin in 1896.

Riemann Zeta Hypothesis

<u>Conjecture</u>: (***Unproven***)
If s is a complex number so that $\zeta(s) = 0$ *then* $\text{Re}(s) = ½$.

What we do know:

- The line x = ½ contains an infinite number of zeroes of $\zeta(s)$.
- The first 70,000,000 or so lie on that line.

Power Series

Definition:

If $\{a_n\}$ is a sequence, we define the series $\sum a_n x^n$ as a *power series* in x. For a given sequence a power series is a function $f(x)$ whose domain consists of those values of x for which the series converges.

Power series behavior is typical to that of the geometric series.

$\sum x^n$ converges for $|x|<1$, so the domain of this function: $f(x) = \sum x^n$ is the open interval (-1,1).

Convergence of Power Series

Proposition:

Suppose that the power series $\sum a_n x^n$ converges for $x = x'$ and diverges for $x = x''$, then $\sum a_n x^n$

1. *converges absolutely for each x satisfying $|x| < |x'|$;*
2. *diverges for each x satisfying $|x| > |x''|$*

Convergence of Power Series

Proof:

1) If $x' = 0$ there is nothing to prove.

Assume $x' \neq 0$. $\sum a_n x'^n$ converges $\Longrightarrow \{a_n x'^n\} \to 0$ $\Longrightarrow \{a_n x'^n\}$ bounded $\Longrightarrow \exists M > 0 \ni |a_n x'^n| \leq M$ for all n.

Let $|x| < |x'|$, then

$$\left|a_n x^n\right| = \left|a_n x'^n\right| \cdot \left|\frac{x}{x'}\right|^n \leq M\left|\frac{x}{x'}\right|^n$$

Convergence of Power Series

If $|x/x'| < 1 \implies \sum M|x/x'|^n$ converges $\implies \sum|a_n x^n|$ converges $\implies \sum a_n x^n$ converges.

2) Assume $|x| > |x''|$ and $\sum a_n x^n$ converges. Then first part $\implies \sum a_n x''^n$ converges, contradicting the given.

Convergence of Power Series

Proposition:

Let $\sum a_n x^n$ be a power series, then one of the following must hold.

1. $\sum a_n x^n$ *converges absolutely for all* x;
2. $\sum a_n x^n$ *converges only at* $x = 0$;
3. *there is a number* $\rho > 0$ *so that* $\sum a_n x^n$ *converges absolutely for* $|x| < \rho$ *and diverges for* $|x| > \rho$.

Convergence of Power Series

Proof:

Let $S = \{x \in \mathbf{R} \mid \sum a_n x^n \text{ converges}\}$. Note: $0 \in S$.

S unbounded $\Longrightarrow$ by previous Proposition, $\sum a_n x^n$ converges absolutely for all x.

Assume S bounded. Let $\rho = \text{lub}\{S\}$, which exists by the Least Upper Bound Axiom.

$x > \rho \Longrightarrow x \notin S \Longrightarrow \sum a_n x^n$ diverges.

Let $x < -\rho$ and let $x < x_0 < -\rho \Longrightarrow \rho < -x_0 < -x$

$\Longrightarrow \sum a_n(-x_0)^n$ diverges $\Longrightarrow \sum a_n x^n$ diverges

Convergence of Power Series

Suppose $\exists\, x'' \ni |x''| < \rho$ and $\sum a_n x''^n$ diverges.

Then $\sum a_n x^n$ diverges for all $x > |x''| \Longrightarrow |x''|$ is an upper bound for S.

This contradicts $\rho = \text{lub}\{S\}$. ■

ρ is the *radius of convergence* for the power series.

$S = [-\rho, \rho], (-\rho, \rho], [-\rho, \rho)$, or $(-\rho, \rho)$ and all can occur. This is called the *interval of convergence*. In Case 1, $\rho = \infty$ and $S = \mathbf{R}$; Case 2, $\rho=0$ and $S=\{0\}$.

Ratio Test for Power Series

Proposition:

Let $\sum a_n x^n$ be a power series with radius of convergence ρ. If $a_n \neq 0$ for all n and

$$\{a_{n+1}/a_n\} \to q$$

then

1. $\rho = \infty$ *if* $q = 0$;
2. $\rho = 0$ *if* $q = \infty$;
3. $\rho = 1/|q|$ *otherwise.*

Proof

Problem 1

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+4)} \qquad a_n = \frac{1}{n(n+4)}, \quad a_{n+1} = \frac{1}{(n+1)(n+5)}$$

$$\frac{a_{n+1}}{a_n} = \frac{1}{(n+1)(n+5)} \cdot \frac{n(n+4)}{1} = \frac{n(n+4)}{(n+1)(n+5)}$$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n(n+4)}{(n+1)(n+5)} = 1 = q$$

$$\rho = \frac{1}{q} = 1$$

Problem 1

$$\sum_{n=1}^{\infty} \frac{x^n}{n(n+4)}$$

$$\rho = 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{n(n+4)}$$

converges by comparison with $\sum 1/n^2$.

$$\rho = -1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+4)}$$

converges by Alternating Series Test.

$$S = [-1, 1]$$

Problem 2

$$\sum_{n=0}^{\infty}\frac{n^n}{n!}x^n \qquad a_n=\frac{n^n}{n!},\quad a_{n+1}=\frac{(n+1)^{n+1}}{(n+1)!}$$

$$\frac{a_{n+1}}{a_n}=\frac{(n+1)^{n+1}}{(n+1)!}\cdot\frac{n!}{n^n}=\frac{n!}{(n+1)!}\cdot\frac{(n+1)^{n+1}}{n^n}=\frac{1}{n+1}\cdot\frac{(n+1)^{n+1}}{n^n}$$

$$=\frac{(n+1)^n}{n^n}=\left(\frac{n+1}{n}\right)^n=\left(1+\frac{1}{n}\right)^n$$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e=q$$

$$\rho=\frac{1}{e}$$

Trigonometric Series

Definition:

Let $\{a_n\}$ and $\{b_n\}$ be sequences, we say that $a_n = O(b_n)$ if

$$\lim_{n\to\infty}\frac{a_n}{b_n} \text{ exists.}$$

$$\sqrt[3]{n^3+n+1} = O(n)$$

$$\frac{3n+5}{4n^4-5n^2+6} = O(n^{-3})$$

Trigonometric Series

Proposition:

Let $\{a_n\}$ be a sequence so that $a_n = O(n^p)$ for some $p < -1$. Then the series $\sum a_n \cos(nx)$ and $\sum a_n \sin(nx)$ both converge absolutely for all x.

Proof:

$a_n = O(n^p) \Longrightarrow \{a_n/n^p\}$ converges $\Longrightarrow \{a_n/n^p\}$ bounded, so $|a_n/n^p| \leq M$ for all $n \Longrightarrow$

$|a_n \cos(nx)| \leq |a_n| \leq Mn^p \Longrightarrow \sum a_n \cos(nx)$ converges absolutely.